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On the Polyhedral Set-Invariance Conditions for Time-Delay Systems

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Abstract:
In this paper the concept of set invariance for time-delay systems is introduced with a specific attention to the linear discrete-time case. We are interested in the definition of a $\mathcal{D}$*(elay)*-invariant set with respect to a bounded polyhedral subset of the state-space. $\mathcal{D}$-invariance conditions are derived based on the Minkowski addition in a first stage, and subsequently translated in feasibility-based tests in order to obtain an efficient computation time.

Keywords: Set Invariance, Constrained Systems, Time-Delay Systems.

1. INTRODUCTION

The *invariant set theory* is an important topic in mathematics and engineering, receiving an increased attention in control literature related to constrained control systems or robust control design (see for instance the monograph Blanchini and Miani (2008), the survey paper Blanchini (1999) and the references therein). In this paper we are interested in the polyhedral invariant sets (Bitsoris, 1988a). Even if the complexity of this kind of representation is higher than in the ellipsoidal case (Kurzhanski and Valyi, 1998), polyhedral sets have the advantage to follow accurately the shape of the limit (maximal/minimal) invariant sets in different frameworks (Artstein and Rakovic, 2008).

Delay Systems represent a class of systems for which the reaction to exogenous signals is not instantaneous. Propagation and transport phenomena, communication, heredity and competition in population dynamics are examples of time-delay systems. Various motivating examples and related discussions ca be found in Niculescu (2001); Gu et al. (2003); Michiels and Niculescu (2007). The concept of set-invariance for time-delay systems is difficult to characterize. To the best of the authors knowledge, there are few references to this problem in the literature. In Dambrine et al. (1995); Goubet-Bartholomeus et al. (1997) the existence conditions for set invariance of continuous time-delay systems are derived using the same arguments as Bitsoris (1988b,a). In the framework of nonlinear model predictive control for time-delay systems, Esfanjani et al. (2009) obtain terminal invariant regions using ellipsoidal sets.

In the discrete-time case, set invariance for time-delay systems has been addressed in Olaru and Niculescu (2008); Lombardi et al. (2009a,b); Gielen et al. (2010). It was shown that for a system affected by delays can be modeled as an uncertain polytopic system. A stabilizing feedback gain and an invariant set can be obtained in an extended state-space framework, where all the delayed control entries (or states) must be stored. As the dimension of the augmented state-space depends on the delay and sampling period, it can lead to complicated polyhedral sets, making the problem intractable. In order to avoid this inconvenient, Lombardi et al. (2010) proposed a stabilizing method on the original state-space dimension, based on Lyapunov-Krasovskii candidates, but the invariant set treatment is still performed in the extended state space.

The present paper concentrates on set-invariance properties of polyhedral sets for discrete time-delay systems in a *non-augmented state space* framework. The concept of $\mathcal{D}$-invariance, introduced in this paper, can be understood as a set-invariance in both current and retarded (delayed) states. It is shown that the computationally expensive $\mathcal{D}$-invariance verification method based on Minkowski addition can be avoided by reducing this problem to a feasibility problem (and its dual form), related to the half space representation of the polyhedral sets.

It is worth mentioning that the present work is connected to the positive invariance of polyhedral sets with respect to multivariable discrete-time systems described by ARMA models (Vassilaki and Bitsoris, 1999). In the same time, part of the invariance conditions for time-delay systems presented here join the results obtained upon the extended Farkas' Lemma in Hennet and Tarbouriech (1998).

Basic notions and definitions: Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^*$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}^*$ denote the field of real numbers, the set of non-negative reals, the set of nonzero real numbers, the set of integer numbers, the set of non-negative integers and the set of nonzero integer numbers, respectively. We denote $\mathbb{R}^n$ a Euclidean space and $(\mathbb{R}^n)^d := \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ the d-times cross product of Euclidean spaces. For every subset Π of $\mathbb{R}$ we define $\mathbb{R}_\Pi := \{k \in \mathbb{R} \mid k \in \Pi\}$ and $\mathbb{Z}_\Pi := \{k \in \mathbb{Z} \mid k \in \Pi\}$. For an arbitrary number $x \in \mathbb{R}$, $|x|$ denotes its absolute value. For a matrix $A \in \mathbb{R}^{r\times s}$, $\{a_{i,j}\} \in \mathbb{R}$ denotes the i^{th} row and j^{th} column element, for $i \in \mathbb{Z}_{[1,r]}$ and $j \in \mathbb{Z}_{[1,s]}$. For a matrix A, $\text{diag}(A, k)$ denotes a diagonal matrix with k matrices A on the main diagonal and zeros elsewhere. I_n denotes the identity matrix of dimension $n \times n$ and $\mathbb{1}$ denotes a vector of appropriated dimensions containing exclusively ones. For a vector $x \in \mathbb{R}^n$ let $\|x\|$ denote its Euclidean norm and $\|x\|_\infty$ denotes its infinity norm, i.e. $\|x\|_\infty = \max_{j\in\mathbb{Z}_{[1,n]}} \{|x_j|\}$, where x_j is the j^{th} element of x. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. For two arbitrary sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}^n$

$$\mathcal{A} \oplus \mathcal{B} = \{x + y \mid x \in \mathcal{A}, y \in \mathcal{B}\}$$

denotes their Minkowski sum. Given a sequence of subsets of $\mathbb{R}^n$, i.e. $\{\mathcal{A}_i\}_{i\in\mathbb{Z}_{[a,b]}}$ with $a \in \mathbb{Z}_+$ and $b \in \mathbb{Z}_{\geq a}$, we define $\bigoplus_{i=a}^{b} A_i := A_a \oplus \cdots \oplus A_b$. For an arbitrary matrix $A \in \mathbb{R}^{m\times n}$ and a set $\mathcal{P} \subseteq \mathbb{R}^n$, we define:

$$A\mathcal{P} = \{y \in \mathbb{R}^m | y = Ax,\ x \in \mathcal{P}\}.$$

For a non-empty closed convex set $\mathcal{P} \in \mathbb{R}^n$, the support function $S(\mathcal{P}, .) : \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$S(\mathcal{P}, u) = \sup_{x\in\mathcal{P}} \langle x, u\rangle \tag{1}$$

where $\langle ., .\rangle$ denotes the inner product on $\mathbb{R}^n$.

2. PRELIMINARIES ON SET INVARIANCE

Consider the discrete-time autonomous system:

$$x(k + 1) = f(x(k)), \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state vector at the time $k \in \mathbb{Z}_+$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

Definition 2.1. Let $\varepsilon \in \mathbb{R}_{[0,1)}$. A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called *contractive* with respect to system (2) if:

$$f(\mathcal{P}) \subseteq \varepsilon\mathcal{P}. \tag{3}$$

For $\varepsilon = 1$, $\mathcal{P}$ is called an *invariant* set with respect to (2). □

The next result shows that invariance property is linked to the classical notion of Lyapunov stability (Hahn, 1967).

Proposition 2.2. If $V(x)$ is a Lyapunov function for the dynamical system (2), then the set $\mathcal{N}(V, c) = \{x : V(x) \leq c\}$ is an invariant set with respect to the same dynamics.

Definition 2.3. (Blanchini, 1995) Consider a convex and compact polyhedral set containing the origin:

$$\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq w\},$$

with $F \in \mathbb{R}^{r\times n}$, $w \in \mathbb{R}^r$. The polyhedral function associated to $\mathcal{P}$ is called a Minkowski function:

$$V(x) = \max_{j\in\mathbb{Z}_{[1,r]}} \{\max\{\{(Fx)_j\}, 0\}\}.$$

where $\{(Fx)_j\}$ denotes the j^{th} element of Fx. This function can be seen as a vector infinity-norm (Kiendl et al., 1992; Loskot et al., 1998):

$$V(x) = \|\max\{Fx, 0\}\|_\infty.$$

Remark 2.4. The Minkowski function of a set $\mathcal{P}$ can be used as polyhedral Lyapunov candidate for stability analysis of dynamical systems upon the Lyapunov stability theorem (Blanchini, 1995).

Remark 2.5. The Definition 2.3 is stated for general polyhedral sets $\mathcal{P}$ containing the origin in their interior. The result holds similarly for symmetric polyhedral sets containing the origin in their interior if:

$$V(x) = \|Fx\|_\infty. \tag{4}$$

□

Proposition 2.6. (Bitsoris, 1988a) The convex polyhedral set:

$$\mathcal{P} = \{x \in \mathbb{R}^n | Fx \leq w\},$$

with $F \in \mathbb{R}^{r\times n}$, $w \in \mathbb{R}^r$, is invariant with respect to

$$x(k + 1) = Ax(k) \tag{5}$$

if there exists a matrix $H \in \mathbb{R}^{r\times r}$ with nonnegative elements such that:

$$FA - HF = 0 \tag{6}$$

$$(H - \mathbb{1})w \leq 0. \tag{7}$$

□

3. DELAY-DIFFERENCE EQUATIONS AND RELATED INVARIANCE DEFINITION

The classical approaches use an extended state-space representation for the treatment of the time-delay systems (Olaru and Niculescu, 2008; Lombardi et al., 2009a,b). In order to avoid this complex framework, we present several tools for alternative set invariance characterization.

Consider a delay-difference equation of the form:

$$x(k + 1) = \sum_{i=0}^{d} A_i x(k - i), \tag{8}$$

where $x(k) \in \mathbb{R}^n$ is the state vector at the time $k \in \mathbb{Z}_+$. $A_i \in \mathbb{R}^{n\times n}$, for all $i \in \mathbb{Z}_{[0,d]}$. We assume that all the initial conditions of system (8) satisfy $x(-i) \in \mathbb{R}^n$, for all $i \in \mathbb{Z}_{[0,d]}$.

The next theorem states the stability conditions for the dynamical system (8).

Theorem 3.1. Consider the Lyapunov-Razumikhin function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that there exist the radially unbounded functions $\phi(\cdot), \omega(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and non-decreasing with $\phi(0) = \omega(0) = 0$ and $\varepsilon \in \mathbb{R}_{[0,1)}$.

Denote $\mathbf{x}(k)^\top = \left[x(k)^\top\ x(k-1)^\top\ \ldots\ x(k-d)^\top\right]^\top \in (\mathbb{R}^n)^{d+1}$.

Consider the function $\tilde{V} : (\mathbb{R}^n)^{d+1} \to \mathbb{R}^n$ with $\tilde{V}(\mathbf{x}(k)) \triangleq \max_{i\in\mathbb{Z}_{[0,d]}} \{V(x(k - i))\}$.

If the following hold:

(i) $\phi(\|x\|) \leq V(x) \leq \omega(\|x\|)$, $\forall x \in \mathbb{R}^n$,
(ii) $V(x(k + 1)) - \varepsilon\tilde{V}(\mathbf{x}(k)) \leq 0$, $\forall k \in \mathbb{Z}_+$, $\forall \mathbf{x}(0) \in (\mathbb{R}^n)^{d+1}$

then the system (8) is globally asymptotically stable.

If $\varepsilon = 1$ the function $V(x(k))$ is called a *weak Lyapunov-Razumikhin function.* Although the existence of a weak