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Generalized aging intensity functions

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ABSTRACT

A family of generalized aging intensity functions is introduced and studied. The functions characterize lifetime distributions of univariate positive absolutely continuous random variables. Further on, the generalized aging intensity orders are defined and analyzed.

1. Introduction

In the paper we study properties of positive unbounded and absolutely continuous random variables with distribution functions F and corresponding density functions $f(x)=\frac{dF(x)}{dx}$ positive on $(0,+\infty)$. In the reliability theory these variables are mainly used to describe elements and systems life. A classic notion of the lifetime analysis is the failure rate function of F (known also as the hazard rate function) which is defined as

$$r_F(x)=\frac{f(x)}{1-F(x)}=-\frac{d\ln[1-F(x)]}{dx}. \tag{1}$$

Other related and popular notions are the cumulative failure rate function (called often shortly hazard function)

$$R_F(x)=\int_0^x r_F(t)dt=-\ln[1-F(x)], \tag{2}$$

and the average failure rate

$$H_F(x)=\frac{1}{x}\int_0^x r_F(t)dt=\frac{1}{x}R_F(x)=-\frac{1}{x}\ln[1-F(x)]. \tag{3}$$

[14] introduced the aging intensity of F defined by

$$L_F(x)=\frac{r_F(x)}{H_F(x)}. \tag{4}$$

Note that the aging intensity function can be also determined by means of the following formulae:

$$L(x)=\frac{x\ r_F(x)}{R_F(x)}=\frac{\frac{d}{dx}\ln[1-F(x)]}{\frac{1}{x}\ln[1-F(x)]}=\frac{-x\ f(x)}{[1-F(x)]\ \ln[1-F(x)]}.$$

According to the definition of survival function $\overline{F}(x)=1-F(x)$, the failure rate (1) can be interpreted as the local infinitesimal conditional probability of an instantaneous failure occurring immediately after the time point x given that the unit has survived until x. The average failure rate (3) can be treated as a global baseline failure rate. Therefore the aging intensity (4) is defined as the ratio of the instantaneous failure rate r_F to the average failure rate H_F and expresses the average aging behavior of the item. It describes the aging property quantitatively: the larger the aging intensity, the stronger the tendency of aging (see [14]).

Moreover, aging intensity $L_F(x)=\frac{\frac{d}{dx}R_F(x)}{\frac{1}{x}R_F(x)}$ can be treated as the elasticity $E_{R_F}(x)$ of the nondecreasing positive cumulative failure rate function (2). The elasticity is an important economic notion, and its thorough discussion can be found in [30]. If function g is differentiable at x and $g(x)\neq 0$, the elasticity of g at x is defined as $E_g(x)=\frac{\frac{d}{dx}g(x)}{\frac{1}{x}g(x)}$. It measures the percentage the function g changes when x changes by a small amount. It can be also treated as the relative accuracy of approximating value of cumulative function $g(x)$ with use of its derivative $\frac{d}{dx}g(x)$. Accordingly, $L_F(x)=E_{R_F}(x)$ measures the percentage the cumulative failure rate function changes (increases) when time x changes (increases) by a small amount.

It is well known (see, for example [2]) that the failure rate r of an absolutely continuous random variable X with support $(0,+\infty)$ uniquely determines its distribution function F as follows:

$$F_r(x)=1-\exp\left(-\int_0^x r(t)dt\right) \quad \text{for} \quad x\in(0,+\infty).$$

The necessary and sufficient conditions on r for being the failure rate of a distribution function is that r is nonnegative, and has infinite integral over $(0,+\infty)$. Similarly, the cumulative failure rate function R and the average failure rate H uniquely determine their distribution function F through the following relationships:

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$$F_R(x) = 1 - \exp(-R(x)) \quad \text{for} \quad x \in (0, +\infty),$$
$$F_H(x) = 1 - \exp(-xH(x)) \quad \text{for} \quad x \in (0, +\infty).$$

For this purpose, $R(x)$ has to increase from 0 at 0 to $+\infty$ at $+\infty$, and clearly $xH(x)$ has to share the property.

Contrary to the unique characterization of the distribution through the failure rate, the cumulative failure rate function and the average failure rate, the aging intensity L characterizes the family of distributions depending on parameter $0 < k < +\infty$.

Theorem 1. *(see [31]) Let L: $(0, +\infty) \to (0, +\infty)$ satisfy the following conditions:*

$$\int_a^b \frac{L(t)}{t}\mathrm{d}t < +\infty = \int_0^a \frac{L(t)}{t}\mathrm{d}t = \int_a^{+\infty} \frac{L(t)}{t}\mathrm{d}t$$

for all $0 < a < b < +\infty$. Then L is an aging intensity function for the family of absolute continuous random variables with support $(0, +\infty)$ and their distribution functions given by the formula

$$F_{L,k}(x) = 1 - \exp\left(-k\,\exp\left(\int_a^x \frac{L(t)}{t}\mathrm{d}t\right)\right) \quad \text{for} \quad x \in (0, +\infty)$$

and for every $k \in (0, +\infty)$ and for some arbitrarily chosen $a \in (0, +\infty)$.

Note that fixing k we determine value of $F_{L,\ k}$ at a, namely $F_{L,k}(a) = 1 - \exp(-k)$. Although in [31] a different parametrization of the family of distributions was presented we claim that the above one is more universal and meaningful in a general context developed here.

In Section 2, we introduce a new family of generalized aging intensity functions, including the above one as a special case, and describe some properties of them. We show in Section 3 that these generalized aging intensities characterize lifetime distributions. It occurs that some generalized aging intensity functions uniquely characterize single distributions, and the others characterize families of distributions dependent on scaling parameters, as in Theorem 1. Some exemplary characterizations are presented in Section 4. In Section 5, we define stochastic orders based on generalized aging intensities, and prove some relations between them. Finally, in Section 6 we indicate applicability of α-generalized aging intensity function for identification of various compound parametric models of lifetime analysis.

2. Generalized aging intensity

Observe that $R_F(x) = (W_0^{-1} \circ F)(x)$, where $W_0(x) = 1 - \exp(-x)$, $x > 0$, is the standard exponential distribution function, and consequently

$$r_F(x) = \frac{\mathrm{d}R_F(x)}{\mathrm{d}x} = \frac{\mathrm{d}(W_0^{-1} \circ F)(x)}{\mathrm{d}x}.$$

[3–5] proposed and studied a generalization of this concept and defined the G-generalized failure rate function for an arbitrary strictly increasing distribution function G with the density g. Under the assumptions, the G-generalized cumulative failure function and the G-generalized failure rate are defined by

$$R_{G,F}(x) = (G^{-1} \circ F)(x),$$
$$r_{G,F}(x) = \frac{\mathrm{d}R_{G,F}(x)}{\mathrm{d}x} = \frac{\mathrm{d}(G^{-1} \circ F)(x)}{\mathrm{d}x} = \frac{f(x)}{g\left[(G^{-1} \circ F)(x)\right]},$$

respectively (see also [27], [11], [10] for further developments). Accordingly, we define the G-generalized aging intensity as

$$L_{G,F}(x) = \frac{x\ r_{G,F}(x)}{R_{G,F}(x)} = \frac{x\ f(x)}{g((G^{-1} \circ F)(x))\ (G^{-1} \circ F)(x)}. \tag{5}$$

In the paper, we restrict ourselves to the analysis of G-generalized aging intensity functions for G belonging to the parametric family of generalized Pareto distributions for which we obtain most intuitive interpretations. We say that X_α follows a generalized Pareto distribution with parameter $\alpha \in \mathbb{R}$ if its distribution function is expressed as

$$W_\alpha(x) = \begin{cases} 1 - (1 - \alpha x)^{\frac{1}{\alpha}} & \text{for} \begin{cases} x > 0, & \alpha < 0, \\ 0 < x < \frac{1}{\alpha}, & \alpha > 0, \end{cases} \\ 1 - \exp(-x) & \text{for} \quad x > 0, \quad \alpha = 0 \end{cases} \tag{6}$$

(see [24]). For negative and positive α, the distribution functions represent Pareto and power random variables, respectively, up to linear transformations. Case $\alpha = 0$ corresponds to the standard exponential distribution introduced above, and represents the limit of W_α as $\alpha \to 0$. The quantile function of W_α is equal to

$$W_\alpha^{-1}(x) = \begin{cases} \frac{1}{\alpha}[1 - (1 - x)^\alpha] & \text{for} \quad 0 < x < 1, \quad \alpha \neq 0, \\ -\ln(1 - x) & \text{for} \quad 0 < x < 1, \quad \alpha = 0. \end{cases} \tag{7}$$

Let F be a distribution function of the univariate absolutely continuous random variable X with its density function $f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$ and support $(0, +\infty)$. Then

$$R_{W_\alpha,F}(x) = (W_\alpha^{-1} \circ F)(x) = \begin{cases} \frac{1}{\alpha}[1 - [1 - F(x)]^\alpha] & \text{for} \quad x > 0, \quad \alpha \neq 0, \\ -\ln[1 - F(x)] & \text{for} \quad x > 0, \quad \alpha = 0, \end{cases} \tag{8}$$

satisfying $(W_\alpha^{-1} \circ F)(0) = 0$, and

$$r_{W_\alpha,F}(x) = \frac{\mathrm{d}(W_\alpha^{-1} \circ F)(x)}{\mathrm{d}x} = [1 - F(x)]^{\alpha-1} f(x) \quad \text{for} \quad x > 0 \tag{9}$$

are the W_α-generalized cumulative failure function, and the W_α-generalized failure rate, respectively. They are further simply called the α-generalized cumulative failure function, and the α-generalized failure rate, and denoted by $R_{\alpha,\ F}$ and $r_{\alpha,\ F}$, respectively.

Analysis of α-generalized failure rates for various α provides more information about variability of lifetime distribution functions. The simplest and more natural one is 1-generalized failure rate which coincides with the density function. The increasing and decreasing density function gives the most rough illustration of the aging tendency of the life random variable at various time moments. Another widely acceptable and more subtle device is the standard 0-generalized failure rate which compares variability of the instantaneous failing tendency expressed by the density value $f(x)$ at time x with the cumulative failure probability $1 - F(x)$ at some moment after x. For instance, a decreasing failure rate of some life distribution on some time interval means that the density $f(x)$ decreases faster than the cumulative survival function $1 - F(x)$ there. If $r_F(x)$ is increasing in some time period, then $f(x)$ decreases slower (it may even increase) than $1 - F(x)$. Studying various α-generalized failure rates enables us to obtain deeper comparisons of variability rates of the density and survival functions. For instance, decrease of the α-generalized failure rate with $\alpha < 0$ is a sharper condition than the standard decreasing failure rate property, and gives more detailed information about the relations between the density and survival functions. The classes of distributions with monotone α-generalized failure rates were considered by [7–9].

Further, the α-generalized average failure rate is equal to

$$H_{\alpha,F}(x) = \frac{1}{x}R_{\alpha,F}(x) = \frac{1}{x}(W_\alpha^{-1} \circ F)(x)$$
$$= \begin{cases} \frac{1}{\alpha x}[1 - [1 - F(x)]^\alpha] & \text{for} \quad x > 0, \quad \alpha \neq 0, \\ -\frac{1}{x}\ln(1 - F(x)) & \text{for} \quad x > 0, \quad \alpha = 0. \end{cases} \tag{10}$$

Finally, the α-generalized aging intensity being the special case of the G-generalized aging intensity (5) can be determined by the formula

$$L_{\alpha,F}(x) = \frac{r_{\alpha,F}(x)}{H_{\alpha,F}(x)} = \frac{\frac{\mathrm{d}}{\mathrm{d}x}(W_\alpha^{-1} \circ F)(x)}{\frac{1}{x}(W_\alpha^{-1} \circ F)(x)}$$
$$= \begin{cases} \frac{\alpha x[1 - F(x)]^{\alpha-1} f(x)}{1 - [1 - F(x)]^\alpha} & \text{for} \quad x > 0, \quad \alpha \neq 0 \\ -\frac{x f(x)}{[1 - F(x)]\ln[1 - F(x)]} & \text{for} \quad x > 0, \quad \alpha = 0. \end{cases} \tag{11}$$