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Sensitivity estimation of failure probability applying line sampling

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ABSTRACT

This contribution presents a framework for calculating a sensitivity measure for problems of computational stochastic mechanics. More specifically, the sensitivity measure considered is the derivative of the failure probability with respect to parameters of the probability distributions (e.g. mean value, standard deviation) associated with the random input quantities of a system's model. The proposed framework is formulated as a post-processing step of Line Sampling, which is a simulation-based method for estimating small failure probabilities. In particular, the proposed framework comprises two different approaches for estimating the sought sensitivity. The application of the proposed framework and comparison of the two aforementioned approaches is discussed through a number of numerical examples. The results obtained indicate that both approaches allow estimating the sought sensitivity measure.

1. Introduction

The application of probability theory has been widely accepted as a means for quantifying the unavoidable effects of uncertainty on the performance of mechanical and structural systems [54]. Thus, the safety of a system can be measured in terms of, e.g. a failure probability. It should be noted that the failure probability can be highly sensitive to the characterization of the uncertainty in the input variables of a system. That is, the failure probability may vary considerably in case the numerical value of a distribution parameter (such as mean value or standard deviation) varies [7,9,39]. Undoubtedly, evaluating the sensitivity of the probability with respect to such parameters is of paramount importance. For example, it can allow pinpointing the most influential parameters of a model [26,29] or perform reliability-based optimization [2,63]. In this context, the objective of this contribution is proposing a framework for evaluating the sensitivity of the failure probability. The sensitivity measure considered herein is the derivative of the failure probability with respect to distribution parameters that describe the uncertainty in the input variables of a model.

Most of the approaches for probability sensitivity estimation with respect to distribution parameters developed so far have been formulated as a post-processing step of an existing strategy for estimating failure probabilities. For example, the estimation of probability sensitivity applying the First- and Second-Order Reliability Methods [17] has been addressed in, e.g. [8,19,20,33]. In these contributions, the sensitivity analysis is closely linked with the gradient of the so-called design point [17] with respect to the distribution parameters. The estimation of the probability sensitivity applying simulation methods such as Monte Carlo [44], Importance Sampling [42,57] and Subset Simulation (which was originally introduced in [5] and further extended in [3,65]) has also been addressed in, e.g. [29,30,43,58,62]. A common feature found in the latter contributions is that the samples generated for estimating the probability are post-processed in order to obtain the sensitivity estimates, thus requiring no additional system (structural) analyses.

The framework for probability sensitivity estimation proposed in this contribution follows a similar scheme when compared to the approaches described above. In particular, the proposed framework is developed as a post-processing of Line Sampling (LS), which is a simulation method introduced in [34] and further extended in [14,15]. It should be noted that LS produces accurate probability estimates for problems which involve a linear, weakly nonlinear or even mildly nonlinear behavior while exhibiting high efficiency when compared to other simulation strategies [52]. The main idea behind LS is estimating the failure probability by assessing the response of the system along *lines* (which are generated randomly in the space of the uncertain input variables). The proposed framework for probability sensitivity estimation is implemented considering two different approaches. The first approach involves calculating the gradient of the function describing the performance of the system at a specific point for each of the lines associated with LS. The second approach involves the estimation of a one-dimensional integral along each of the lines generated when applying LS. Although the two approaches

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are conceptually different, they produce similar sensitivity estimates. The performance of both approaches is discussed using several numerical examples.

It is important to note that the application of LS for probability sensitivity estimation has already been explored in [37,47]. However, the work reported herein possesses substantial differences when compared with those contributions. First, the objective of the current work is estimating probability sensitivity with respect to distribution parameters while [47] focuses on estimating sensitivity with respect to deterministic parameters of the model. Second, the results obtained in the current work generalize the results on sensitivity analysis along each line associated with LS which were presented in [37]. Third, the contributions [37,47] are developed following the first approach of the framework reported herein, while the second approach has not been explored as yet in context with LS.

The range of application of the proposed approach for probability sensitivity estimation based on LS is similar to that of LS applied for probability estimation, i.e. reliability problems that involve weakly to mildly nonlinear behavior. In fact, in such class of problems, LS may exhibit a high numerical efficiency when compared to other approaches for probability estimation, as discussed in [55]. While the proposed approach is also applicable to more general reliability problems, it is expected that its efficiency can decrease. Thus, on one hand, for problems that exhibit a highly nonlinear behavior, the application of Subset Simulation can provide a more efficient means for estimating probability sensitivity, as discussed in [30,58]. On the other hand, for those problems that exhibit a linear or close to linear behavior, the application of the First Order Reliability method may be appropriate [8,19,20,33].

This paper is organized as follows. The formulation of the problem studied in this paper (i.e. sensitivity of the failure probability) is presented in Section 2. Section 3 contains a brief overview on Line Sampling (LS). Two approaches for estimating probability sensitivity applying LS are presented in Sections 4 and 5, respectively. These two approaches are compared in Section 6 while its application to a number of examples is investigated in Section 7. The contribution closes with some conclusions and outlook in Section 8.

2. Formulation of the problem

2.1. Failure probability

Assume that a computational model of a mechanical or structural system of interest is available, which has been generated using an appropriate technique such as, e.g. the finite element method [6]. A total of n input variables of this model are uncertain and are characterized as random variables $X_i,\ i = 1, \dots, n$. The physical values that these input variables may assume are denoted as $x_i,\ i = 1, \dots, n$. For the sake of simplicity and without loss of generality, it is assumed in the remaining part of this contribution that these random variables are independent. However, possible dependencies between these random variables could be accounted for considering appropriate models, see e.g. [36,45]. The probability density function (pdf) associated with each input random variable is denoted as $f_{X_i}(x_i|\boldsymbol{\theta}_i)$, where $\boldsymbol{\theta}_i$ is a vector that collects the distribution parameters of X_i such as mean, standard deviation, etc. The joint pdf is denoted as $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta})$, where $\boldsymbol{x} = [x_1, \dots, x_n]^T$, $\boldsymbol{\theta} = [\boldsymbol{\theta}_1^T, \dots, \boldsymbol{\theta}_n^T]^T$ and $(\cdot)^T$ represents transpose. Due to the independence between random variables, it is evident that $f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) = \prod_{i=1}^n f_{X_i}(x_i|\boldsymbol{\theta}_i)$.

As some of the input variables of the model are random, the response of the model is random as well. Moreover, some particular realizations of the input variables may lead to an undesirable response of the system, such as loss of serviceability or structural collapse. The chances that such undesirable response occur can be measured in terms of a failure probability.

$$p_F = \int_{g_{\boldsymbol{x}}(\boldsymbol{x})\leq 0} f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) d\boldsymbol{x} \tag{1}$$

In the above equation, p_F denotes failure probability and $g_{\boldsymbol{x}}(\boldsymbol{x})$ is the so-called performance function [11,17], which assumes a value equal or smaller than zero whenever a realization $\boldsymbol{x}$ of the random input variables causes an undesirable structural response. Throughout this work, it is assumed that the performance function is differentiable with respect to $\boldsymbol{x}$.

The numerical evaluation of the failure probability integral is usually a challenging task. This stems from two issues. First, the number of random variables involved in a problem may be large, thus precluding the application of numerical quadrature. Second, for most cases of practical interest, the performance function $g_{\boldsymbol{x}}(\boldsymbol{x})$ is not known analytically. In fact, its evaluation must be performed often in a point-wise manner for particular values of $\boldsymbol{x}$, which implies performing a deterministic system (structural) analysis. The two aforementioned issues favor the application of approximate methods (such as the First- and Second-order Reliability Methods [17]) and simulation techniques (such as Monte Carlo and its more advance variants [4]) for evaluating the failure probability.

2.2. Sensitivity of failure probability

The structure of Eq. (1) indicates that the value of the failure probability is affected by the vector $\boldsymbol{\theta}$ that groups the distribution parameters. A possible means for quantifying the sensitivity of the failure probability with respect to these distribution parameters is calculating the partial derivative of p_F with respect to each entry in $\boldsymbol{\theta}$ [62], i.e.:

$$\frac{\partial p_F}{\partial \theta_{l,i}} = \int_{g_{\boldsymbol{x}}(\boldsymbol{x})\leq 0} h_{x_i,\theta_{l,i}}(x_i|\boldsymbol{\theta}_i) f_{\boldsymbol{X}}(\boldsymbol{x}|\boldsymbol{\theta}) d\boldsymbol{x},\ l = 1, \dots, n_i,\ i = 1, \dots, n \tag{2}$$

where $\theta_{l,i}$ represents the lth distribution parameter associated with the ith random variable, n_i is the number of distribution parameters associated with the ith random variable and $h_{x_i,\theta_{l,i}}(x_i|\boldsymbol{\theta}_i)$ is the following function.

$$h_{x_i,\theta_{l,i}}(x_i|\boldsymbol{\theta}_i) = \frac{1}{f_{X_i}(x_i|\boldsymbol{\theta}_i)} \frac{\partial f_{X_i}(x_i|\boldsymbol{\theta}_i)}{\partial \theta_{l,i}} \tag{3}$$

The challenges associated with the calculation of the partial derivative of the probability in Eq. (2) are – in principle– similar to those associated with the failure probability integral in Eq. (1).

2.3. Transformation into standard normal space

A common practice in structural reliability is expressing the failure probability integral in the standard normal space. Thus, each of the input random variables of the model (which are denoted as *physical* random variables) is mapped into a standard normal random variable. In view of the assumption of independence between physical random variables, such projection is performed by equating the cumulative density function associated with the ith physical random variable with the cumulative density function of the ith standard normal random variable [16]. That is, $F_{X_i}(x_i|\boldsymbol{\theta}_i) = \Phi(z_i)$, where $F_{X_i}(\cdot)$ and $\Phi(\cdot)$ are the cumulative distribution functions of the physical random variable X_i and standard normal random variable Z_i, respectively. In this way, it is possible to define a transformation function such that $z_i = t_i(x_i|\boldsymbol{\theta}_i),\ i = 1, \dots, n$, where $t_i(x_i|\boldsymbol{\theta}_i) = \Phi^{-1}\left(F_{X_i}(x_i|\boldsymbol{\theta}_i)\right)$ and $\Phi(\cdot)^{-1}$ represents the standard normal inverse cumulative distribution function. The collection of the n transformation functions (which is actually a vector-valued function) is denoted as $\boldsymbol{z} = \boldsymbol{t}(\boldsymbol{x}|\boldsymbol{\theta})$.

In view of the definitions discussed above and applying a change of variables, Eqs. (1) and (2) can be recast in the standard normal space as:

$$p_F = \int_{g_{\boldsymbol{z}}(\boldsymbol{z}|\boldsymbol{\theta})\leq 0} \phi_n(\boldsymbol{z}) d\boldsymbol{z} \tag{4}$$

$$\frac{\partial p_F}{\partial \theta_{l,i}} = \int_{g_{\boldsymbol{z}}(\boldsymbol{z}|\boldsymbol{\theta})\leq 0} h_{z_i,\theta_{l,i}}(z_i|\boldsymbol{\theta}_i) \phi_n(\boldsymbol{z}) d\boldsymbol{z} \tag{5}$$