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On the extension of Importance Measures to complex components

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ABSTRACT

Importance Measures are indicators of the risk significance of the components of a system. They are widely used in various applications of Probabilistic Safety Analyses, off-line and on-line, in decision making for preventive and corrective purposes, as well as to rank components according to their contribution to the global risk. They are primarily defined for the case the support model is a coherent fault tree and failures of components are described by basic events of this fault tree.

In this article, we study their extension to complex components, i.e. components whose failures are modeled by a gate rather than just a basic event. Although quite natural, such an extension has not received much attention in the literature. We show that it raises a number of problems. The Birnbaum Importance Measure and the notion of Critical States concentrate these difficulties. We present alternative solutions for the extension of these notions. We discuss their respective advantages and drawbacks.

This article gives a new point of view on the mathematical foundations of Importance Measures and helps us to clarify their physical meaning.

1. Introduction

Importance Measures are indicators of the risk significance of the components of a system. They are widely used in various applications of Probabilistic Safety Analyses, off-line and on-line, in decision making for preventive and corrective purposes, as well as to rank components according to their contribution to the global risk. Presentations of these indicators and discussions about their mathematical properties and their physical interpretations can be found for instance in References [1–12].

Importance Measures are primarily defined for the case the support model is a coherent Fault Tree and failures of components are represented by basic events of this fault tree. In this article, we study their extension to complex components, i.e. to components whose failures are modeled by a gate and not just by a Basic Event. Although quite natural, this extension has not received much attention in the literature (see however [13,14]).

We proceed in two steps. First, we revisit definitions of the main Importance Measures and we show that, in the case of simple components, each of them characterizes the probability of a set of minterms, i.e. of a set of global states of the system under study. Namely,

- The states in which both the component and the system are failed, as for the Diagnostic Importance Factor and the Risk Achievement Worth.
- The states in which the system is failed but the component is working, as for the Risk Reduction Worth.
- The Critical States, i.e. states in which failing/repairing the component suffices to repair/fail the system, as for the Birnbaum Importance Measure (also called Marginal Importance Factor) and the Critical Importance Measure.

This new way of defining Importance Measures via minterms does not mean that they need to be calculated via minterms. Calculations can actually be still performed by means of Minimal Cutsets or Binary Decision Diagrams. Its interest stands in the soundness of mathematical definitions, the independence of any calculation means and the simplicity of physical interpretations.

Second, we show that this nice correspondence between the probabilistic definition and the minterm interpretation does not hold for complex components. The Birnbaum Importance Measure and the notion of Critical States concentrate the difficulties.

So far, complex components have been studied in the literature only via the extension of Importance Measures to groups of (simple) components (see e.g. [6,10,12,14–19]). Several authors showed already that the definition of the Birnbaum Importance Measure in terms of a partial derivative is not suitable for groups of components (see e.g. [20,21]). They proposed therefore to define the Birnbaum Importance Measure as the difference between the conditional

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probability that the system is failed given that all components of the group are failed and the conditional probability that the system is failed given that none of the components of the group are failed. This definition is actually equivalent to the partial derivative one in the case the group is reduced to a single component. It could be applied to complex components as well, as proposed for instance by Sutter [14]. We show however that this indicator is much too coarse. First, it does not allow us to distinguish components with different structure functions (a parallel sub-system would be evaluated the same way as a series sub-system). Second, it leads to consider as critical states, states in which the system is failed but the component is working. We show that finer extensions can be defined, but necessarily to the price of losing the correspondence between the probabilistic definition and the minterm interpretation.

This article contributes therefore to establish more firmly the mathematical foundations of Importance Measures and to clarify their physical interpretation. It also gives hints to tool developers about which indicators are worth to calculate from a safety model.

The remainder of this article is organized as follows:

- Section 2 introduces basic definitions and properties. It gives a formal definition for the notion of coherence and Critical States.
- Section 3 revisits definitions and interpretations of Importance Measures in the case the support model is a coherent Fault Tree and failures of components are represented by Basic Events.
- Section 4 discusses extensions of Importance Measures to complex components and groups of components.
- Finally, Section 5 concludes the article.

2. Basic definitions and properties

2.1. Boolean formulas and minterms

Throughout this article we consider Boolean formulae (Fault Trees) built over a denumerable set $\mathcal{E}$ of variables and the usual connectives "$\cdot$" (and), '$+$'" (or) and "$-$" (not). Variables are also called Basic Events.

We use uppercase letters E, A, B, C, possibly with subscripts, to denote Basic Events.

We use lowercase letters s, t, c, possibly with subscripts, to denote Boolean formulae. We denote by $\mathrm{var}(s)$ the variables occurring in the formula s.

Let s be a Boolean formula. A variable assignment of s is a function from $\mathrm{var}(s)$ into $\{0,1\}$ (0 and 1 stand respectively for False and True). Variable assignments are lifted-up as usual into functions from formulae into $\{0,1\}$ using the truth tables of connectives. A Boolean formula s is interpreted as the Boolean function $[\![s]\!]$, i.e. as the function from variable assignments of s into $0,1$, defined as follows: for any variable assignment σ of s, $[\![s]\!](\sigma)=1$ if $\sigma(s)=1$ and 0 otherwise.

In this paper, we do not need to distinguish between syntax and semantics. Therefore, we shall assimilate the Boolean formula s with its semantics $[\![s]\!]$.

A literal is either a variable E or its negation $\overline{E}$. We use uppercase letters L, I, J, possibly with subscripts to denote literals. We denote by $\overline{L}$ the opposite of a literal L, given that $\overline{\overline{L}} \equiv L$. Let $\mathcal{L}$ be a set of literals, we denote by $\overline{\mathcal{L}}$ the set of negations of literals of $\mathcal{L}$, i.e. $\{\overline{L}; L \in \mathcal{L}\}$.

A product is a conjunct of literals that does not contain both a variable and its negation. Let s be a Boolean formula. A minterm of s is a product that contains a literal built over each variable of $\mathrm{var}(s)$. We use lowercase Greek letters $\sigma, \tau\, \pi$, and ρ, possibly with subscripts, to denote products and minterms. We denote as $\mathrm{Minterms}(\mathcal{E})$ the set of $2^{|\mathcal{E}|}$ minterms that can be built over a set of Basic Events $\mathcal{E}$.

There is a one-to-one correspondence between variable assignments and minterms (and therefore between Boolean functions and sets or sums of minterms): the variable assignment σ one-to-one corresponds with the minterm π such that π contains the positive literal E if $\sigma(E)=1$ and the negative literal $\overline{E}$ if $\sigma(E)=0$. It follows that any Boolean formula s is equivalent to the set of minterms π such that $\pi(s)=1$. Minterms of $\mathrm{Minterms}(\mathcal{E})$ are the atoms of the Boolean algebra built over $\mathcal{E}$.

For the sake of the convenience, we shall use a set theory notation, i.e. we shall note $\pi \in s$ when $\pi(s)=1$ and $\pi \notin s$ when $\pi(s)=0$. Note also that $\pi \notin s$ if and only if $\pi \in \overline{s}$.

Example (Minterms). As an illustration, consider the two formulae $s_1 = A\cdot B + A\cdot C + B\cdot C$ and $s_2 = A\cdot B + \overline{A}\cdot C$. The minterms of s_1 and s_2 are as follows:

$$s_1 \equiv A\cdot B\cdot C + A\cdot B\cdot \overline{C} + A\cdot \overline{B}\cdot C + \overline{A}\cdot B\cdot C$$
$$s_2 \equiv A\cdot B\cdot C + A\cdot B\cdot \overline{C} + \overline{A}\cdot B\cdot C + \overline{A}\cdot \overline{B}\cdot C$$

From a more practical perspective, assuming that the plant under study is modeled by a Fault Tree, minterms just describe full state vectors of the plant. Let s be the formula associated with the Top-Event of a fault tree, then products π that satisfy s ($\pi \in s$) are the cutsets of s.

Let s be a Boolean function and $\mathcal{L} = \{L_1, \ldots, L_k\}$ be a set of literals built over a subset of $\mathrm{var}(s)$. We denote by $s|\mathcal{L}$ the Boolean function built over $\mathrm{var}(s) \ll \mathrm{var}(\mathcal{L})$ as follows:

$$s|\{L_1, \ldots, L_k\} \stackrel{\text{def}}{=} \{\pi \,|\, L_1.\ldots.L_k \cdot \pi \in s\}$$

For the sake of the simplicity, we write $s|L_1, \ldots, L_k$ (instead of $s|\{L_1, \ldots, L_k\}$). The notation $s|L$ is intentionally close to the one used for conditional probabilities because it is really what it means: s given L.

Example ($s|L$)**.** Considering the function s_1 defined above, the following equalities hold:

$$s_1|A \equiv B\cdot C + B\cdot \overline{C} + \overline{B}\cdot C = B + C$$
$$s_1|\overline{A} \equiv B\cdot C$$
$$s_1|A, B \equiv C$$
$$s_1|A, \overline{B} \equiv C$$

2.2. Shannon decomposition and coherence

We can now state the Shannon decomposition.

Property 1 (*(Logical) Shannon decomposition*). *Let s be a Boolean formula and E a Basic Event of* $\mathrm{var}(s)$*. Then, the following equivalence holds*:

$$s \equiv E\cdot s|E + \overline{E}\cdot s|\overline{E}$$

Throughout this article, we shall assume that Basic Events are independent from a statistical viewpoint. The above equivalence is translated in terms of probability by the either of the two equalities that will play an important role latter.

Property 2 (*(Probabilistic) Shannon decomposition*). *Let s be a Boolean formula and E a Basic Event of* $\mathrm{var}(s)$*. Then, the following equalities hold*:

$$\Pr\{s\} = \Pr\{E\}\cdot \Pr\{s|E\} + [1-\Pr\{E\}]\cdot \Pr\{s|\overline{E}\} \quad (1)$$

$$\Pr\{s\} = \Pr\{E\}\cdot [\Pr\{s|E\} - \Pr\{s|\overline{E}\}] + \Pr\{s|\overline{E}\} \quad (2)$$