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Assessment of the transition-rates importance of Markovian systems at steady state using the unscented transformation

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ABSTRACT

The Unscented Transformation (UT) is a technique to understand and compute how the uncertainty of a set of random variables, with known mean and variance is propagated on the outputs of a model, through a reduced set of model evaluations as compared with other approaches (e.g., Monte Carlo). This computational effort reduction along with the definition of a proper UT model allows proposing an alternative approach to quantify the transition rates (TR) having the highest contribution to the variance of the steady-state probability, for each possible state of a system represented by a Markov model. The so called "main effects" of each transition rate, as well as high order component interactions are efficiently derived from the solution of only $(2n+1)$ linear system of simultaneous equations, being n the number of transition rates in the model.

1. Introduction

Markov models are often used to assess the performance of repairable systems when considering reliability and/or availability metrics [1–5]. The assessment generally involves the calculations of the steady-state probabilities for each possible system performance state by solving a series of simultaneous equations based on transition rates among component states. A system analyst could then understand how the transition rates affect the performance of a system the most.

One difficult problem is the quantification of these transition rates; if they are available, e.g. from measurement, they may only be considered as estimates with their corresponding uncertainty. Several approaches based on different techniques have been considered to assess the effects of uncertainties in Markov models, for example: Markov set-chains theory [6]; Linear programming [7]; Imprecise theory [8]; Interval probabilities [9,10]; partial derivatives [11,12], the Monte Carlo (MC) method [5], Interval Arithmetic (IA) [13,14], and Affine Arithmetic (AA) [15].

In general, the interval widths derived using these approaches can be used as an importance measure to rank transition rates individually, by quantifying their effects on steady-state probabilities for each possible system performance state. However, they do not allow assessing the possible effects of interactions among components' transition rates (e.g., the simultaneously effect of two or more transition rates on steady-state probabilities).

Rocco and Zio [16] proposed the use of global sensitivity analysis (SA) as an alternative approach for assessing transition-rate importance in Markov models. These global methods evaluate the effect of a factor while all others factors are varying as well, thus allowing the exploration of the multi-dimensional input space [17]. The uncertainty in the transition rates is considered as a random variable modeled via a known probability density function and then, specific SA techniques such as FAST [18] are applied to evaluate their corresponding importance. However, one of the main disadvantages of such methods is that they require many evaluations (i.e., solving several linear systems of simultaneous equations).

To avoid the high computational cost, Rocco and Zio [19] proposed the use of a special meta-model based on polynomial chaos expansion (PCE) techniques. A PCE is a multi-dimensional polynomial approximation of the model with coefficients determined by evaluating the model in a significantly reduced set (when compared against traditional SA techniques) of predetermined points. Importance index values are then derived directly from the PCE.

Rocco and Ramirez-Marquez [20] proposed obtaining the importance of the components in the reliability assessment of a system, using an extension of the Unscented Transformation (UT) technique. The approach requires evaluating a very small set of models, linearly proportional to the number of components. In

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addition, the UT considers that model variables could be statistically depended (e.g., due to the presence of an external variable that simultaneously affects a set of transition rates).

This paper extends the approach presented in Rocco and Ramirez Marquez [20] for assessing the importance of the transition rate uncertainties in the evaluation of the steady-state probability of Markovian behaved systems. To our knowledge, this assessment has not been previously analyzed.

The remainder of this paper is organized as follows: Section 2 defines the Markov model to be considered. Section 3 reviews the UT approach while Section 4 describes the approach to obtain importance measures. Section 5 provides results of experimentation and Section 6 presents the main conclusions.

1.1. Acronyms, Notations and Assumptions

1.1.1. Acronyms

CTMC	: continuous time Markov chain
IM	: importance measure
pdf	: probability density function
SA	: sensitivity analysis
SR	: state reduction algorithm

1.1.2. Notation

λ_{ij}	transition rate from state i to state j.
m	number of state
Q	transition rate matrix
n	number of different transition rates
π	steady-state probability vector (π_1, π_2, … π_m)
π_i	steady-state probability of state i
S_i	main order sensitivity IM
S_{Ti}	total order sensitivity IM

1.1.3. Assumptions

1. The system is in steady-state.
2. Time between two successive failures and service time each follows an exponential distribution, so failure rate and repair rate are constant; the process is stationary.
3. The number of states in the Markov models is finite.
4. The Markov chain is aperiodic and irreducible.

2. Markov model

The transition of a system through different states that represent different operational components states can be described by a discrete-state, continuous time Markov chain (CTMC) $Z=\{z(t), t\geq 0\}$, with finite state space $E=\{1,2,\ldots, m\}$. For each $i,j\in E$, let λ_{ij} be the transition rate from state i to state j and $\lambda_{ii}=-\sum_{i\neq j}\lambda_{ij}$, with λ_{ii} representing the principal diagonal of matrix Q, and defined as the negative sum of transitional rates from j into i. Where Q is defined as the $m\times m$ transition rate matrix and $P_i(t)$ be the probability that the system is in state i at time t. $P_i(t)$ are obtained by solving the matrix differential equation

$$\dot{P}(t)=P(t)Q$$

given the initial conditions for $P(0)$.

Let $\pi=(\pi_1, \pi_2, \ldots, \pi_m)$ be the steady-state vector of Z. π can be found by solving the following linear system [1]:

$$\begin{cases} \pi Q=0 \\ \sum_{i=1}^{m}\pi i=1 \end{cases} \tag{1}$$

Associated with each state of the CMTC is a performance level of the system (or a reward rate). For example, in a two-state model, one state may be related to the operating state with full capacity and the other to a failed state with null capacity. States with the same level of performance could be combined. The probability of the combined states is defined as the sum of the probability of the states to be combined.

Since the numerical values of π_i depend on the values of the elements of the transition rate matrix Q, it is clear that their variations, due to uncertainty, will affect the value of probability π_i.

Fig. 1 shows the state space diagram for a one component repairable system [1], where Λ is the failure rate and μ is the repair rate.

The transitional rate matrix is

$$Q=\begin{pmatrix} -\Lambda & \Lambda \\ \mu & -\mu \end{pmatrix} \tag{2}$$

From (1) and (2) the steady-state probabilities π_i, can be obtained by solving the following linear systems:

$$\begin{aligned} -\Lambda\pi_1+\mu\pi_2&=0 \\ \pi_1+\pi_2&=1 \end{aligned} \tag{3}$$

The solution produces $\pi_1=\mu/(\Lambda+\mu)$ and $\pi_2=\Lambda/(\Lambda+\mu)$ that is, each π_is are non-linear functions of the transition rates. If Λ and/or μ associated uncertainty is modeled as random variable then the steady-state probability π_{ii}s also a random variable. Theoretically the Q matrix is defined by (m^2-m) transition rates, so the number of factors to be varied is large, even for moderate values of m. This value assumes that there are transition rates among all the states. In general the number of different transition rates is smaller due to some modeling assumptions, such as the existence of equal components or no-common cause transitions, among others. In this paper the number of different transition rates is defined as n.

The set of linear Eq. (1) could be solved by a classical decomposition method (LDU) or by a Gauss–Seidel iterative method [21]. However, in this paper the State Reduction (SR) approach proposed by Kumar and Billinton [22] is used. Their approach consists on a stable algorithm to calculate steady-state probability using a method that modifies the state transition matrix, reducing one state per iteration, until the Markov system reduces to a 2-state model. This method does not involve any subtraction so the errors due to rounding or cancellation are minimized.

3. The unscented transformation

The UT uses the fact that it is “easier to approximate a probability distribution than to approximate an arbitrary non-linear function or transformation” [23].The approach is based on

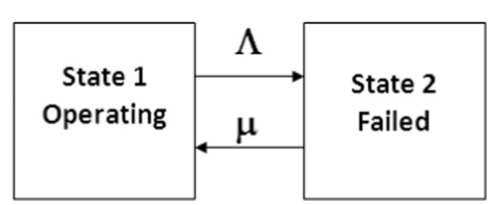

Fig. 1. State space diagram for single component repairable system.