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Time and cost constrained optimal designs of constant-stress and step-stress accelerated life tests

David Han *

Department of Management Science and Statistics, University of Texas at San Antonio, Texas 78249, USA

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ABSTRACT

By running life tests at higher stress levels than normal operating conditions, accelerated life testing quickly yields information on the lifetime distribution of a test unit. The lifetime at the design stress is then estimated through extrapolation using a regression model. To conduct an accelerated life test efficiently with constrained resources in practice, several decision variables such as the sample allocation proportions and the stress durations should be determined carefully at the design stage. These decision variables affect not only the experimental cost but also the estimate precision of the lifetime parameters of interest. In this work, under the constraint that the total experimental cost does not exceed a pre-specified budget, the optimal decision variables are determined based on $C/D/A$-optimality criteria. In particular, the constant-stress and step-stress accelerated life tests are considered with the exponential failure data under time constraint as well. We illustrate the proposed methods using two real case studies, and under the identical budget constraint, the efficiencies of these two stress loading schemes are compared in terms of the ratio of optimal objective functions based on the information matrix.

1. Introduction

With increasing reliability and substantially long life-spans of products, it is often difficult for standard life testing methods under normal operating conditions to obtain sufficient information about the failure time distribution of the products. This difficulty is overcome by accelerated life test (ALT) where the test units are subjected to higher stress levels than normal for rapid failures. By applying more severe stresses, ALT collects information on the parameters of lifetime distributions more quickly. The lifetime at the design stress is then estimated through extrapolation using a regression model. Some key references in the area of ALT are Nelson [25], Meeker and Escobar [20], and Bagdonavicius and Nikulin [2].

In order to conduct ALT efficiently with constrained resources in practice, several decision variables such as the sample allocation proportions and the stress durations should be determined carefully at the design stage. These decision variables affect not only the experimental cost but also the estimate precision of the lifetime parameters of interest. For this reason, the optimal ALT design has attracted great attention in the reliability literature. Miller and Nelson [22] initiated research in this area by considering a simple step-stress model with exponential failure time distribution under complete sampling. The fundamental model used was the one proposed by Sedyakin [27], which is known as the *cumulative exposure model*. This model was further discussed and generalized by Bagdonavicius [1] and Nelson [24]. Nelson and Kielpinski [26] studied the optimally censored ALT for normal and lognormal distributions. By minimizing the asymptotic variance of the maximum likelihood estimator (MLE) of the acceleration factor, Bai et al. [4] determined the optimal stress change time point of partially ALT under lognormal lifetime distribution. By minimizing the asymptotic variance of a lot acceptability statistic, Bai et al. [3] also designed the sampling plans for failure-censored ALT under Weibull distribution subject to the expected test time constraint. The optimum combinations of multiple stresses for ALT were investigated by Elsayed and Zhang [7] based on the proportional hazards (PH) model. Recently, Monroe et al. [23] used the generalized linear model (GLM) to generate the optimal design matrix for planning ALT in relation to other factors such as censoring and a nonlinear response function while Yang and Pan [30] proposed the optimal ALT plans under interval censoring using the PH model with GLM.

A Bayesian approach to parameter estimation was presented by Erto and Giorgio [8] for the Weibull-inverse power law model reducing the required number of failures based on a priori engineers' knowledge. Van Dorp and Mazzuchi [29] on the other hand devised a general Bayesian inferential method for ALT under Weibull lifetimes where a multivariate prior distribution was indirectly defined for scale parameters at various stress levels. Meeter and Meeker [21] then developed the statistical models and ALT plans with a non-constant shape parameter. Recently, exact

* Tel.: +1 210 458 7895.
E-mail address: david.han@utsa.edu

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conditional inference for a step-stress model with exponential competing risks was studied by Balakrishnan and Han [5], Han and Balakrishnan [10]. Gouno et al. [9], Balakrishnan and Han [6] discussed the problem of determining the optimal stress duration under progressive Type-I censoring; see also Han et al. [11] for some related comments. Under complete sampling, Hu et al. [13] studied the statistical equivalency of a simple step-stress ALT to other stress loading designs while Han and Ng [12] compared the efficiencies of general k-level constant-stress and step-stress ALT under complete sampling and Type-I censoring.

Most of these studies, however, only addressed issues related to the statistical estimation precision. In practice, constraints for conducting ALT, in particular the cost constraint, can be a major consideration besides obtaining accurate and relevant information about the products. Tang and Xu [28] considered planning a constant-stress ALT with the objective of meeting a desired precision target for statistical estimation along with a cost target. Liao [16] discussed a three-level compromise constant-stress ALT for estimating the long-run cost rate of a periodical replacement policy under the fixed normal operating condition. Recently, Zhang and Liao [31] considered an ALT design with the objective of improving the statistical and energy efficiencies based on the product reliability and the physical characteristics of the test equipment. Others considered planning ALT under some implicit cost constraints through the use of different censoring times at different stress levels, expected test time constraint, and by obtaining the optimal sample sizes although they do not always satisfy the budget constraints.

Here we explicitly formulate the cost model which is a function of the ALT design components. Under the constraint that the total experimental cost does not exceed a pre-specified budget, we then develop a framework for generating the ALT plans to achieve the optimal statistical precision of the estimates of interest. In particular, the general k-level constant-stress and step-stress ALT are considered with the exponential lifetime distribution for units subjected to stress under Type-I censoring. It is assumed that the relationship between the mean lifetime parameter and stress level is log-linear along with the accelerated failure time (AFT) model for the effect of changing stress in step-stress ALT. Then, the optimal design variables, allocation proportions and stress durations, are determined under various optimality criteria. Naturally, this framework includes some of the existing approaches as its special cases simply by tuning the parameters involved. The proposed methods are illustrated using two real case studies, and under the identical budget constraint, the efficiencies of these two stress loading schemes are compared using the ratio of optimal objective functions based on the information matrix as a measure of relative efficiency.

The rest of the paper is organized as follows. Section 2 presents the model description and formulation, derives the MLEs of the model parameters and the associated Fisher information for k-level constant-stress ALT and step-stress ALT. The cost functions for the constrained optimal designs are defined in Section 3. Section 4 then defines the three optimality criteria based on the Fisher information (*viz.*, variance, determinant, and trace) and talks about the existence of optimal design points in each case under Type-I censoring. Section 5 illustrates the proposed methods using two real datasets and discusses the relative efficiency of these two classes of ALT under consideration. Finally, Section 6 is devoted to some concluding remarks and future works in this direction.

2. Model description, MLE and Fisher information

Let $s(t)$ be the given stress loading (a deterministic function of time) for ALT. Let us also assume that there exists an upper bound of stress level, s_H, below which the failure mode is the same as the one under the normal use-stress level, s_U. That is, the applied stress should not be so high that the underlying failure mechanism changes. The standardized stress loading is then defined as

$$x(t) = \frac{s(t) - s_U}{s_H - s_U}, \quad t \geq 0$$

so that the range of $x(t)$ is $[0,1]$. Depending on its shape, $x(t)$ can model many commonly applied stress loading schemes as special cases including constant-stress, (modified) step-stress, cyclic-stress, ramp-stress (progressive-stress), and any combination of these. Now, let us define $0 \equiv x_0 \leq x_1 < x_2 < \cdots < x_k \leq 1$ to be the ordered standardized stress levels to be used in the test. As the basis of constructing an ALT model, it is further assumed that under any specific stress level x_i, the exponential distribution describes the failure mechanism of a test unit. That is, the probability density function (PDF) and the corresponding cumulative distribution function (CDF) of the lifetime of a test unit at stress level x_i are

$$f_i(t) = \frac{1}{\theta_i}\exp\left(-\frac{t}{\theta_i}\right), \quad 0 < t < \infty, \tag{1}$$

$$F_i(t) = 1 - S_i(t) = 1 - \exp\left(-\frac{t}{\theta_i}\right), \quad 0 < t < \infty, \tag{2}$$

respectively. Also, it is assumed that under any stress level x_i, the mean time to failure (MTTF) of a test unit, θ_i, is a log-linear function of stress given by

$$\log \theta_i = \alpha + \beta x_i, \tag{3}$$

where the regression parameters α and β are unknown and need to be estimated. The log-linear relationship is a commonly used and well-studied model for the accelerated exponential distribution model. Along with its simplicity, the log-linear link represents several life–stress relationships built from physical principles such as Arrhenius, inverse power law, Eyring, temperature-humidity, and temperature-non-thermal; see Miller and Nelson [22], Kececioglu and Jacks [14].

Here we consider two popular classes of ALT: constant-stress and step-stress. In constant-stress testing, a unit is tested at a fixed stress level until failure occurs or the life test is terminated, whichever comes first. On the other hand, (step-up) step-stress testing allows the experimenter to gradually increase the stress levels at some pre-fixed time points during the test. The following subsections present the MLEs of α, β and the associated Fisher information for general k-level constant-stress ALT and step-stress ALT. No notational distinction is made in this article between the random variables and their corresponding realizations. Also, we adopt the usual conventions that $\sum_{j=m}^{m-1} a_j \equiv 0$ and $\prod_{j=m}^{m-1} a_j \equiv 1$.

2.1. k-level constant-stress test under Type-I censoring

A constant-stress test under Type-I censoring proceeds as follows. For $i = 1, 2, \ldots, k$, $N_i = \Upsilon(n\pi_i)$ units are allocated on test at stress level x_i , where $\Upsilon(\cdot)$ is a discretizing function of one's choice, mapping its argument to a non-negative integer. To ensure $\Upsilon(n\pi_i) \approx n\pi_i$, $\Upsilon(\cdot)$ could be one of *round*$(\cdot)$, *floor*$(\cdot)$, *ceiling*$(\cdot)$ and *trunc*$(\cdot)$, for example. Since the above definition of N_i complicates the distributional derivation of associated random quantities, for simplicity, we shall assume in all subsequent derivations that $N_i \equiv n\pi_i$ such that $\sum_{i=1}^{k} N_i = n$ or equivalently, $\sum_{i=1}^{k} \pi_i = 1$. $\pi_i = N_i/n$ is the allocation proportion of units (out of total n units under the test) assigned to stress level x_i. The allocated units are then tested until time τ_i at which point all the surviving items are withdrawn, thereby terminating the life test. Let n_i denote the number of units failed at stress level x_i in time interval $[0, \tau_i)$ and $y_{i,l}$ denote the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \ldots, n_i$ while $N_i - n_i$ denotes the number of units censored at time τ_i. Obviously, when there is no right censoring (*viz.*, $\tau_i = \infty$