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Imprecise inference for warranty contract analysis

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ABSTRACT

This paper presents an investigation into generalised Bayesian analysis of warranty contracts, using sets of prior distributions within the theory of imprecise probability. Explicit expressions are derived for optimal lower and upper bounds for the expected profit for the manufacturer of a product, corresponding to an imprecise negative binomial model for which two sets of prior distributions are studied. The results can be used to set a maximum value of compensation such that the manufacturer's expected profit remains positive, under vague prior knowledge.

1. Introduction

Warranties are important aspects of many contracts between consumers and manufacturers. Typically, decisions about such contracts must be made at an early stage, when the available knowledge about the product reliability might be vague. While the Bayesian approach is attractive to investigate warranties, meaningfully assigning a single prior distribution might be difficult and it might not fully reflect available information. In particular, if one attempts to model lack of prior information, the generalised Bayesian approach using theory of imprecise probability, in which sets of prior distributions are used instead of a single prior distribution, provides an attractive framework for inference that can be used to analyse warranty contracts.

An introduction to general theory of imprecise probability has been presented by Augustin et al. [1], while an earlier detailed mathematical introduction to such theory was presented by Walley [2]. Introductions and overviews of imprecise probability with specific attention to topics in reliability and risk have been presented by the current authors [3–5]. The problem studied in this paper concerns a basic model for warranties, proposed by Singpurwalla [6] and also mentioned by Aven [7]. It does not include detailed analysis of real-world warranty data, which is an important and challenging topic which could benefit from analysis with the use of statistical methods based on imprecise probabilities. Recent contributions to statistical methods for analysis of real-world warranty data, including many further references, have been presented by Wu [8] and Gupta et al. [9]. Standard Bayesian analysis of warranty claim data has been proposed by many authors, for example, Stephens and Crowder [10], Chen and Popova [11], Wu and Huang [12], and Akbarov and Wu [13].

Section 2 introduces the basic setting for the analysis of warranty contracts considered in this paper. Section 3 presents a standard Bayesian approach for such an analysis, which is generalised through the use of an imprecise probability model in Section 4. While this model is closely related to popular imprecise probability models, it has a quite obvious disadvantage which is addressed in Section 5, effectively by using a restricted set of prior distributions. The main results presented in this paper are explicit expressions for the lower and upper expected profits for the manufacturer with a specified warranty contract. These are optimal lower and upper bounds and they enable valuation of compensation under this contract in order for the expected profit to remain positive. The presented imprecise probability models only assume vague prior knowledge and explicitly reflect this through these lower and upper bounds. The results are illustrated by examples in the respective sections. The paper ends with some concluding remarks in Section 6. Detailed proofs of the propositions in this paper are presented in an appendix.

2. Warranty contract analysis

Consider a scheme of typical warranty contracts as proposed by Singpurwalla [6] and also considered by Aven [7]. The scheme models the exchange of items from a large collection of similar items between a manufacturer (seller A) and a consumer (buyer B).

Let n be the number of items that the buyer would like to purchase. These items are supposed to be exchangeable with

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regard to their intended functioning and to have independent and identically distributed (iid) failure behaviour. Each item is required to be used for τ units of time, so the iid assumption for their lifetimes implies that each item meets this requirement with the same probability and the random success of any item in doing so is independent of that of other items, conditional upon the value of this probability. Throughout this paper, and in line with common practice, the probability of an item functioning successfully over the period considered is assumed to be high, so failures are relatively rare. It is assumed that an item can only fail once.

Suppose that the buyer B is willing to pay x monetary units, say dollars, per item, and is prepared to tolerate at most a total of z failures for all the n units in the time interval $[0,\tau]$. For each failure in excess of z, the buyer B needs to be compensated at the rate of y dollars per item. In effect, the quantity τ can be viewed as the duration of a warranty. One of the questions of interest is determination of the maximum compensation y per item in order for the seller to keep a non-negative expected profit.

Suppose that it costs c dollars to produce a single unit of the item, then the sale of n units at price x leads to income $n(x-c)$ dollars for seller A. If the buyer B experiences z or fewer failures in $[0,\tau]$, then this income is equal to A's profit. However, if B experiences $i>z$ failures in $[0,\tau]$ then A's liability is $(i-z)y$ leading to total profit of $n(x-c)-(i-z)y$ dollars.

Formally, the number of failing items in the given time period of length τ should be modelled by a Binomial distribution. However, due to the reasonable assumption that failures during this period are relatively rare, it is common practice [6] to use the Poisson distribution as approximation, this simplifies computation and is assumed henceforth in this paper. In this model, the parameter reflecting the quality of the items is the failure rate λ, which represents the average number of failing items among n items during a unit time interval. Let $p(i|\lambda)$ denote the probability for the event that exactly i items will fail during the time interval $[0,\tau]$. For known value of the parameter λ, this probability is

$$p(i|\lambda)=\frac{(\lambda\tau)^i\exp(-\lambda\tau)}{i!}. \tag{1}$$

Note that, while these probabilities are positive for all integers $i\geq 0$, the assumption that items will only fail quite rarely implies that $p(i|\lambda)$ for $i>n$ will be neglectably small, hence the approximation mentioned above remains reasonable. The corresponding *expected profit* for seller A, denoted by G (for 'gain'), for known value of λ, is

$$\mathbb{E}_\lambda G=n(x-c)-y\sum_{i=z+1}^{n}(i-z)p(i|\lambda). \tag{2}$$

In this paper, the scenario considered is that seller A will aim at non-negative expected profit, so $\mathbb{E}_\lambda G\geq 0$. Of course, this could be replaced with a different target for the expected profit, the mathematical analysis would be easily adapted and is not discussed further. If the seller has strong background information concerning the failure rate λ, it may be possible to consider it to be known. However, in many applications such information is not available. The Bayesian approach, reviewed in the following section, is the standard method for dealing with a not fully known failure rate.

3. Standard Bayesian approach

If the parameter λ is unknown, it can be considered as a random quantity for which a probability density function $\pi(\lambda|\boldsymbol{\theta})$ can be assumed. In this case, the Bayesian approach can be applied for computing the expected profit, which is determined as the unconditional expected value

$$\mathbb{E}G=\int_\Omega \mathbb{E}_\lambda G\cdot\pi(\lambda|\boldsymbol{\theta})\,\mathrm{d}\lambda=n(x-c)-y\sum_{i=z+1}^{n}(i-z)\int_\Omega p(i|\lambda)\cdot\pi(\lambda|\boldsymbol{\theta})\,\mathrm{d}\lambda.$$

Here $\boldsymbol{\theta}$ is the vector of parameters of π and $\Omega=\mathbb{R}_+$ is the set of possible values of λ. The corresponding probability of exactly k failures occurring during a period of length τ is

$$P(k)=\int_\Omega p(k|\lambda)\cdot\pi(\lambda|\boldsymbol{\theta})\,\mathrm{d}\lambda.$$

The Bayesian approach enables prior information, mainly based on expert judgement, to be combined with data. Suppose that the prior distribution $\pi(\lambda|\theta)$ reflects the expert's opinion about the possible values for λ prior to collecting any information. Suppose that data become available of the following form: n items have been tested for m periods, which can be of variable length. Suppose that the number of failing items, out of n, during period $j\in\{1,\dots,m\}$ is k_j, and that the length of this period is τ_j. It should be noted that a more general scenario, with numbers of items being tested during the different periods not being equal to n, is quite straightforward to analyse following a similar setting but with the parameter λ explicitly related to a single item; this is left as an exercise for the reader, the current restriction simplifies the presentation and does not really limit the model with regard to the main new results as presented in the following sections.

It is convenient in Bayesian analysis to choose a prior distribution such that resulting computations in order to derive the posterior distribution are easy, which particularly occurs when a conjugate prior distribution is used. This leads to a posterior distribution belonging to the same parametric family of distributions as the prior distribution [14]. The Gamma distribution is a conjugate prior for the parameter λ of the Poisson distribution. Its parameters are $\theta=(a,b)$, with $a>0$, $b>0$, and it has the probability density function

$$\pi(\lambda|a,b)=\Gamma(a,b)=\frac{1}{\Gamma(a)}b^a\lambda^{a-1}\exp(-b\lambda),\ \lambda>0.$$

where $\Gamma(a)$ is the gamma function.

Suppose that data become available for n items during m time periods, as described above, and let $K=k_1+\dots+k_m$ and $T=\tau_1+\dots+\tau_m$. The corresponding posterior predictive probability for the event that, out of n further items, k will fail during a time period length τ can be derived by standard Bayesian methods [14]. These probabilities, for $k\geq 0$, are given by the Negative Binomial distribution and are equal to

$$P(k)=\frac{\Gamma(a+K+k)}{\Gamma(a+K)k!}\left(\frac{b+T}{b+T+\tau}\right)^{a+K}\left(\frac{\tau}{b+T+\tau}\right)^{k}. \tag{3}$$

Note again that these $P(k)$ are positive for all $k\geq 0$, but with relatively few items failing these probabilities for $k>n$ will be extremely small, ensuring that the approximate model does not lead to complications.

Returning to the warranty model analysed in this paper, as introduced in Section 2, the expected profit for the manufacturer is equal to

$$\mathbb{E}G=n(x-c)-y\sum_{k=z+1}^{n}\frac{(k-z)\Gamma(a+K+k)}{\Gamma(a+K)k!}\left(\frac{b+T}{b+T+\tau}\right)^{a+K}\left(\frac{\tau}{b+T+\tau}\right)^{k}. \tag{4}$$

This standard Bayesian scenario is illustrated by Example 1.

Example 1. Suppose a buyer is considering to purchase $n=100$ items at a cost of $x=20$ dollars per item. Suppose that it costs $c=16$ to produce each single item. Let the time period considered be of length $\tau=1$ and suppose that the buyer would be willing to