Reliability Engineering and System Safety 134 (2015) 110–117

Contents lists available at ScienceDirect

Reliability Engineering and System Safety

journal homepage: www.elsevier.com/locate/ress

Sectional global sensitivity measures

S. Pannier*, W. Graf

Institute for Structural Analysis, Technische Universität Dresden, 01062 Dresden, Germany

ARTICLE INFO

Article history:
Received 3 August 2013
Received in revised form
27 July 2014
Accepted 1 September 2014
Available online 8 September 2014

Keywords:
Sectional global sensitivity measures
Importance factor
Global sensitivity measure
Derivative based global sensitivity measure
Regional sensitivity measure

ABSTRACT

In this paper the approach of sectional sensitivity measures is introduced. Opposite to well-known global sensitivity measures not only a singleton value is provided to appraise the functional input–output interrelation but rather a more detailed description of these interrelation is enabled. Therefore, the domain of definition (input space) and/or the codomain (result space) are subdivided in a finite number of subdomains/subranges. The evaluation of global sensitivity measures in these subdomains/subranges allows for a proper appraisal of the functional interrelation in local regions.

The theoretic background of sectional sensitivity measures is elaborated in detail and exemplified by means of analytical functions. The advantages of sectional sensitivity measures are discussed by means of a medical intervention planning of a radio frequency ablation.

1. Introduction

The simulation based design process of engineering structures is a complex and ambitious task, especially, if multiple input parameters have to be handled. Versatile tools are at hand to optimize the structure or assess the reliability. But most often, engineers long for data mining tools to understand functional input–output interrelations on the basis of some predetermined point sets. Especially the assessment of sensitivities provides the engineer with the desired information [1].

Common sensitivity assessments base on regression analysis, correlation evaluations or "one-at-a-time" investigations [1]. A reasonable application of such measures succeeds only for linear or at least monotonic functions. In contrast global sensitivity measures (GSM) have the potential to capture non-linear input–output interrelations. Thereby, it is essential to specify the domain of definition properly. Several global sensitivity measures are available, see inter alia [2,3], while the two most recognized ones are shortly described in the following. First, a function can be decomposed and the resulting (decomposed) functions are assessed with variance measures. This is denoted as ANOVA decomposition. A comparison of the variances allows for the assessment of input–output interrelations. Such measures are denoted as variance-based global sensitivity measures, see e.g. [1,4–8]. Second, the expectation of function gradients can be assessed. This idea was used, e.g., in [9], and further elaborated in [10–12]. Both approaches of global sensitivity measures can be denoted as state of the art, since they are available in commercial software tools, e.g., [13,14].

As a result of a global sensitivity assessment a singleton sensitivity value is obtained to characterize the influence of each input parameter on the output. All information about the complex (non-linear) functional input–output interrelation is condensed in this single value. Such approaches are reasonable to assess problems with a high number of input parameters, but frequently more detailed information is preferable to deduce decisions. Therefore, the approach of sectional sensitivity measures is presented in this paper. The main idea is to partition the domains of definition or the codomain in a specific manner into subdomains and determine (global) sensitivity measures for each subdomain. This makes a local or regional characterization of functional input–output interrelations possible.

The approach of sectional global sensitivity measures (SGSM) provides a reasonable alternative to visualization tools for input–output interrelations like scatterplots or metamodel (response surface approximation) views. These visualization tools plot individual input–output graphs of high dimensional functions, while the explanatory power of such graphs is distorted by other sensitive input parameters. Such a distortion is avoided for SGSM on account of evaluating global sensitivity measures.

Example. The shortcoming of visualization tools can be demonstrated by means of the Rosenbrock function:

$$a(x,y) = (1-x)^2 + 100 \cdot (y-x^2)^2, \quad x \in [-2,2], \ \ y \in [-1,3]. \tag{1}$$

In Fig. 1 the function a is plotted for both x and y, while the respective other input parameter is fixed.

The characteristic of a varies in dependence of the fixed input parameter considerably. Even though, a is only a two dimensional

* Corresponding author. Tel.: +49 351 46334172; fax: +49 351 46337086.
E-mail addresses: stephan.pannier@tu-dresden.de (S. Pannier), wolfgang.graf@tu-dresden.de (W. Graf).

http://dx.doi.org/10.1016/j.ress.2014.09.009
0951-8320/

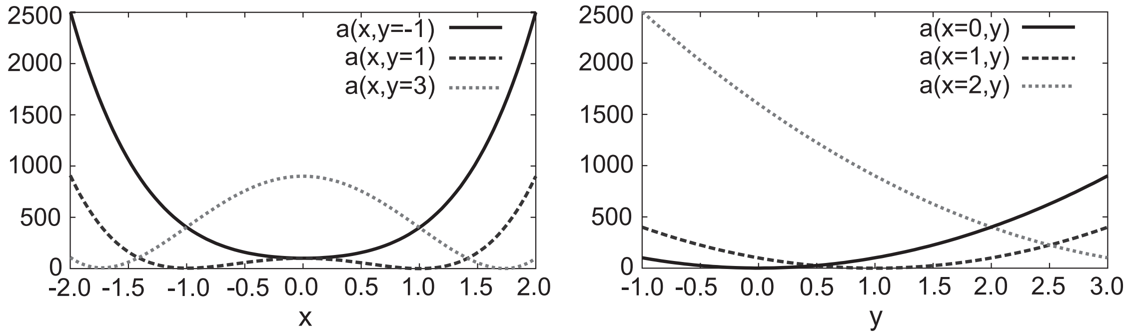

Fig. 1. Plot of function a (see Eq. (1)) for individual input parameters.

function, the evaluation of the functional input–output interrelation in local regions is cumbersome.

The idea of getting sensitivity statements in subregions of the domain of definition was also followed in [15–17]. In these publications the term "regional sensitivity measures" was coined. Thereby, a "reduced range" method is used based on differences between model output mean/variances or Borgonovo's delta indices, see for more details [15–17]. In this approach it is focused on differences in gradients instead. This is comparable to differences between variance based GSM and derivative based GSM.

In general, a computation of SGSM with a sufficient accuracy is hindered most often, if finite element methods are applied. This is caused by either the high computational effort of a single run or the unavailability of derivatives to determine global sensitivity measures. In order to preserve the applicability metamodels are used, see e.g. [18–20]. Here, artificial neural networks are applied.

In this paper a short introduction to GSM is given in Section 2. Then, the approach of sectional sensitivity is introduced in Section 3. Finally, in Section 4 the features of SGSM are demonstrated by means of analytical functions and the applicability is shown by means of a radio frequency ablation.

2. Definitions of global sensitivity measures

A differentiable function:

$$f : H \subset \mathbb{R}^n \to B \subset \mathbb{R} \tag{2}$$

is given. In a sensitivity analysis the impact of an *individual* input parameter $x_i \in \mathbb{R}$ on an output $f(x) = f(x_i, x_{\sim i}) \in B$ in comparison to *all other* input parameters $x_{\sim i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$ has to be assessed. A sensitivity measure is intended to express the impact of x_i with a single number $S_i \in [0,1]$. A (normalized) sensitivity measure S_i is defined as

$$S_i = \frac{\hat{S}_i}{\sum_{j=1}^{n} \hat{S}_j}, \tag{3}$$

where $\hat{S}_i$ is a measure for a specific characteristic of f with regard to x_i. The quality of a sensitivity statement depends mainly on the definition of $\hat{S}_i$. In general, numerous definitions are available highlighting different characteristics of f. The presented approach of sectional sensitivity measures applies derivative-based GSM. The definition of derivative-based GSM bases on the evaluation of the expectation of partial derivatives. Therefore, different approaches are discussed. In first numerical approaches based on elementary effects proposed in [9] the plain "gradients" were assessed, while in [10,11] it is emphasized, that a change of sign will distort the results. Alternatively, the application of the absolute partial derivative $|\partial f/\partial x_i|$ is recommended. Thus, one possible way of measuring the sensitivity $\hat{S}_i$ is using

$$G_i = \frac{1}{V(H)} \int_H \left| \frac{\partial f}{\partial x_i}(x) \right| \mathrm{d}x. \tag{4}$$

H is the domain of definition of function f (Eq. (2)) and, thus, of $|\partial f/\partial x_i|$. The factor $V(H)$ measures the volume of H and is defined by $V(H) = \int_H \mathrm{d}x$. If $G_i = 0$, then x_i has no impact on $f(x_i, x_{\sim i})$.

Example. The difference of applying either $\partial f/\partial x_i$ or $|\partial f/\partial x_i|$ is demonstrated by means of the function $a(x,y) = x + y^2$, $x, y \in [-1, 1]$. The expectation of $\partial a/\partial y$ is zero, although the impact of y is not negligible. If, instead, the absolute partial derivative $|\partial a/\partial y|$ is used, this error is avoided since the expectation of $|\partial a/\partial y|$ is two.

In [12] it is suggested to use the square of a partial derivative $(\partial f/\partial x_i)^2$. On the basis of this approach a proportionality to global variance-based sensitivity measures can be shown. Based on [11] an interrelation between the expectation

$$G_i^* = \frac{1}{V(H)} \int_H \frac{\partial f}{\partial x_i}(x)\, \mathrm{d}x. \tag{5}$$

and the variance

$$D_i = \frac{1}{V(H)} \int_H \left(\frac{\partial f}{\partial x_i}(x) - G_i^* \right)^2 \mathrm{d}x. \tag{6}$$

of $\partial f/\partial x_i$ can be expressed with

$$\frac{1}{V(H)} \int_H \left(\frac{\partial f}{\partial x_i}(x) \right)^2 \mathrm{d}x = (G_i^*)^2 + D_i. \tag{7}$$

In [10] it is pointed out, that additional information may be provided by analyzing the variance D_i (see Eq. (6)). Either no or a linear functional input–output interrelation is indicated by $D_i = 0$. With $D_i > 0$ either the degree of nonlinearity or the interrelation of input parameters x_i, x_j $(j \neq i)$ is assessed.

As mentioned before, numerous measures $\hat{S}_i$ are available to assess different characteristics of f. For instance, applying the second moment of $\partial f/\partial x_i$ (Eq. (7)) instead of the second central moment (Eq. (6)) gives different figures. This is shortly demonstrated for the function $a(x,y) = x^2 + y^2$, $x \in [0,1]$, $y \in [1,2]$. The sensitivity evaluated on the basis of Eq. (6) reads $S_x = 0.5$ and $S_y = 0.5$ while the sensitivity determined with Eq. (7) is $S_x = 0.125$ and $S_y = 0.875$. In comparison with variance and derivative based GSM ($S_x = 0.105$, $S_y = 0.895$ and $S_x = 0.25, S_y = 0.75$) the result from Eq. (7) looks more reasonable even though the result from Eq. (6) reveals some useful information about f if the specific definition is known. For application it has to be decided which level of knowledge can be expected from the user.

In the following, the partial derivative sensitivity measures are determined with $\hat{S}_i = G_i$ as given in Eq. (4).