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Gamma lifetimes and one-shot device testing analysis

N. Balakrishnan [a], M.H. Ling [b,*]

[a] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1
[b] Department of Mathematics and Information and Technology, The Hong Kong Institute of Education, Hong Kong SAR, China

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ABSTRACT

Gamma distribution is widely used to model lifetime data in reliability and survival analysis. In the context of one-shot device testing, encountered commonly in testing devices such as munitions, rockets, and automobile air-bags, either left- or right-censored data are collected instead of actual lifetimes of the devices under test. The destructive nature of one-shot devices makes it difficult to collect sufficient lifetime information on the devices. For this reason, accelerated life-tests are commonly used in which the test devices are subjected to conditions in excess of its normal use-condition in order to induce more failures, so as to obtain more lifetime information within a relatively short period of time. In this paper, we discuss the analysis of one-shot device testing data under accelerated life-tests based on gamma distribution. Both scale and shape parameters of the gamma distribution are related to stress factors through log–linear link functions. Since lifetimes of devices under this test are censored, the EM algorithm is developed here for the estimation of the model parameters. The inference on the reliability at a specific mission time as well as on the mean lifetime of the devices is also developed. Moreover, by using missing information principle, the asymptotic variance–covariance matrix of the maximum likelihood estimates under the EM framework is determined, and is then used to construct asymptotic confidence intervals for the parameters of interest. For the reliability at a specific mission time and also for the mean lifetime of the devices, transformation approaches are proposed for the construction of confidence intervals. These confidence intervals are then compared through a simulation study in terms of coverage probabilities and average widths. Recommendations are then made for an appropriate approach for the construction of confidence intervals for different sample sizes and different levels of reliability. A distance-based statistic is suggested for testing the validity of the model to an observed data. Finally, since current status data with covariates in survival analysis and one-shot device testing data with stress factors in reliability analysis share the same data structure, a real data from a toxicological study is used to illustrate the developed methods.

1. Introduction

In a reliability study of battery [1], battery data collected from destructive life-tests were analyzed. Statistical inference for one-shot devices, namely electro-explosive devices, space shuttles, military weapons, and automobile air bags, has also been developed recently. In one-shot device testing data, due to the destructive nature of one-shot devices, lifetime of the devices is always either left- or right-censored, and exact failure times of the devices cannot be observed. One can only observe the condition of the device at a specific inspection time. Collecting sufficient lifetime information about the devices within a limited time for prediction of the reliability at a specific mission time at normal operating conditions becomes quite difficult in this case. For this reason, accelerated life-tests are often used by applying higher levels of stress to the devices in order to induce rapid failure. Then, based on a presumed distributional model and a life-stress relationship, the reliability at a specific mission time at normal operating conditions can be predicted from the data collected at the higher stress levels.

A number of models have been discussed in the literature for one-shot device testing data under accelerated life-tests. Fan et al. [2] adopted the Bayesian approach with priors for studying the reliability and the mean lifetime of electro-explosive devices at normal temperature under the exponential distribution. Subsequently, this approach was compared to the expectation–maximization (EM) algorithm for maximum likelihood estimates by Balakrishnan and Ling [3]. Recently, their results have been generalized in [4] from a single-stress relationship model to a multiple-stress relationship model.

Gamma distribution is commonly used for fitting lifetime data in reliability and survival studies due to its flexibility. Its hazard function can be increasing, decreasing, and constant. When the hazard function of gamma distribution is a constant, it corresponds to the exponential distribution. In addition to the exponential distribution, the gamma distribution also includes the chi-square distribution as a

* Corresponding author.
E-mail address: amhling@ied.edu.hk (M.H. Ling).

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special case. The gamma distribution has found a number of applications in different fields. To quote a few, Husak et al. [5] used it to describe monthly rainfall in Africa for the management of water and agricultural resources, as well as food reserves. Kwon and Frangopol [6] assessed and predicted bridge fatigue reliabilities of two existing bridges, the Neville Island Bridge and the Birmingham Bridge, based on long-term monitoring data. They made use of log–normal, Weibull, and gamma distributions to estimate the mean and standard deviation of the stress range. Tseng et al. [7] proposed an optimal step-stress accelerated degradation testing plan for assessing the lifetime distribution of products with longer lifetime based on a gamma process.

The EM algorithm is a widely used method for computing maximum likelihood estimates (MLEs) of the model parameters in the presence of missing data, while the choice of starting values may be more important under least-squares or Fisher scoring methods. Balakrishnan and Ling [8] showed that there are very few cases of divergence for the EM algorithm when the starting value is far from the true parameter. The monotonicity of the likelihood and convergence properties of the algorithm have been discussed by Dempster et al. [9]. Interested readers may also refer to the book by McLachlan and Krishnan [10] for details on various modifications and extensions of this algorithm. Furthermore, Louis [11] suggested a method of finding the information matrix when the EM algorithm is implemented, using which asymptotic confidence intervals can be constructed for the model parameters. Several different applications of the EM algorithm in different scenarios have been mentioned in [3,4]. In this paper, we develop the EM algorithm for the analysis of one-shot device testing data based on the gamma distribution. Multiple-stress relationship on both scale and shape parameters of the gamma distribution is assumed through log–linear link functions.

To asses the fit of a model to an observed data, a suitable test statistic (with corresponding p-value) is required. In this regard, Balakrishnan and Ling [4] proposed a distance-based statistic for assessing the fit of the model to observed data, and approximated the corresponding p-value through the parametric bootstrap procedure, which becomes quite burdensome computationally as the sample size increases. Here, we derive the p-value of the test statistic by the use of binomial distribution.

The rest of this paper is organized as follows. Section 2 describes the form of the one-shot device testing data under accelerated life-tests based on the gamma distribution and the corresponding likelihood function. In Section 3, the EM algorithm is developed for determining the MLEs of the model parameters, as well as the MLEs of the reliability at a specific mission time and the mean lifetime at normal operating conditions. In Section 4, the derivation of the asymptotic variance–covariance matrix based on the observed information matrix and the asymptotic (asy) confidence intervals are presented. Logistic (logit-), hyperbolic arcsecant (arsent-) and log-transformations are also introduced for the construction of confidence intervals for the reliability at a specific mission time and the mean lifetime of the devices at normal operating conditions. Section 5 describes a goodness-of-fit test and the determination of the corresponding p-value. In Section 6, a simulation study is carried out for evaluating the performance of the proposed methods of inference for different levels of reliability and different sample sizes. Section 7 illustrates an application of the developed method to a study of tumors in mice induced by benzidine dihydrochloride. Section 8 finally provides some concluding remarks.

2. Model formulation and likelihood function

In accelerated life-tests for one-shot devices, the devices are placed in I testing groups. Suppose, for $i=1,2,\ldots,I$, K_i devices are subjected to J types of stress factors, $\{x_{ij}, j=1,2,\ldots,J\}$, under inspection time IT_i. In the i-th test group, number of failures, n_i, is collected. The data thus observed can be summarized as in Table 1.

Table 1
Data on oneshot device testing at various stress levels collected at different inspection times.

Testing group	Inspection time	Numbers of tested devices	Numbers of failures	*Covariates*		
				Stress 1	⋯	Stress J
1	IT_1	K_1	n_1	x_{11}	⋯	x_{1J}
2	IT_2	K_2	n_2	x_{21}	⋯	x_{2J}
⋮	⋮	⋮	⋮	⋮		⋮
I	IT_I	K_I	n_I	x_{I1}	⋯	x_{IJ}

Let T be a gamma random variable with shape parameter $\alpha>0$ and scale parameter $\beta>0$. The probability density function (pdf) and the cumulative distribution function (cdf) of T are given by

$$f(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}}\exp\left(-\frac{t}{\beta}\right),\quad t>0, \tag{1}$$

and

$$F(t)=\int_0^t \frac{y^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}}\exp\left(-\frac{y}{\beta}\right)dy,\quad t>0, \tag{2}$$

respectively. The cdf $F(t)$ can be readily expressed in terms of the lower incomplete gamma ratio as

$$F(t)=\int_0^{t/\beta}\frac{y^{\alpha-1}}{\Gamma(\alpha)}\exp(-y)\,dy=\gamma\left(\alpha,\frac{t}{\beta}\right). \tag{3}$$

Within each testing group, we assume that both shape and scale parameters are related to the stress factors in log–linear forms as

$$\alpha_i=\exp\left(\sum_{j=0}^{J}a_j x_{ij}\right) \tag{4}$$

and

$$\beta_i=\exp\left(\sum_{j=0}^{J}b_j x_{ij}\right), \tag{5}$$

where $x_{i0}=1$ for all i. For notational convenience, we denote $\boldsymbol{z}=\{IT_i,K_i,n_i,i=1,2,\ldots,I\}$ for the observed data, and $\boldsymbol{\theta}=\{a_j,b_j,j=0,1,\ldots,J\}$ for the model parameters. The observed likelihood function is then given by

$$L(\boldsymbol{\theta};\boldsymbol{z})\propto\prod_{i=1}^{I}[F(IT_i;\alpha_i,\beta_i)]^{n_i}[1-F(IT_i;\alpha_i,\beta_i)]^{K_i-n_i}. \tag{6}$$

3. EM algorithm

The EM algorithm is a powerful technique for finding the MLEs of the model parameters in the presence of missing data; see McLachlan and Krishnan [10]. It is suitable for the determination of the MLEs of the model parameters when all the failure times of the devices are censored like in the case of one-shot device testing data. It simply involves approximating the missing data (the expectation step/E-step) and maximizing the corresponding likelihood function (the maximization step/M-step) in each iteration. These are developed here in this section for the problem under consideration.