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Assessing parameter uncertainty on coupled models using minimum information methods

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ABSTRACT

Probabilistic inversion is used to take expert uncertainty assessments about observable model outputs and build from them a distribution on the model parameters that captures the uncertainty expressed by the experts. In this paper we look at ways to use minimum information methods to do this, focussing in particular on the problem of ensuring consistency between expert assessments about differing variables, either as outputs from a single model or potentially as outputs along a chain of models. The paper shows how such a problem can be structured and then illustrates the method with two examples; one involving failure rates of equipment in series systems and the other atmospheric dispersion and deposition.

1. Introduction

An important element of Probabilistic Risk Analysis is assessment of uncertainty in model outputs. Physical models do not perfectly represent the phenomena they are meant to describe for many reasons: lack of complete understanding of the physical phenomena, deliberate simplifications in the model (for example, because of the need to run the model quickly), inadequate choice of model parameters, and so on.

Bedford and Cooke [1] stress the importance of assessing uncertainties for observable quantities when using probability. Since it is typically model outputs that are observable quantities, and many model parameters are not directly observable (maybe having no direct physical interpretation) it is therefore necessary to consider ways of taking probability distributions that describe the uncertainty in model output quantities and “back-fitting” these to generate a distribution on the model parameters so that this matches the uncertainty specified for the output parameters in the following sense: If we randomly choose a set of model parameters and compute the model outputs then those outputs are also random. The distribution of the outputs obtained is called the push-forward of the distribution on the model parameters and should match the uncertainty for the observable quantities.

The problem of Probabilistic Inversion (PI) is simply the problem of computing a distribution on the model parameters with the property that its push forward matches that specified for the distribution on the observable quantities. See Bedford and Kraan [2] and Kurowicka and Cooke [3] for different approaches to this problem. Having defined an uncertainty distribution on the model parameters, this then allows us to make predictions, incorporating our uncertainty, on model outputs for any set of model inputs.

The probabilistic inversion problem is typically either under- or over-specified. This means that the constraints imposed by the output distributions are either not sufficiently strong to lead to a single solution to the PI problem or they are mutually contradictory, and give rise to an infeasible problem.

Previous approaches have used minimally informative distributions to solve the problem of under-specification, and have either ignored the problem of infeasibility or used slightly ad hoc approaches to deal with it. This paper develops the ideas proposed in Bedford [4] to use minimum information methods [5,6] to provide guidance in the specification of constraints that are not infeasible. It also gives a method to map out the feasible region for the constraints. Minimum information methods use constraints on expected values rather than quantile information, though we discuss how quantile information can be incorporated.

The paper generalizes the context given above to one of the “coupled” models, that is, the situation where we have several models which are sequentially linked in the sense that the outputs of the first are the inputs of the second, and so on. Hence specifications may be made of distributions of input and output parameters, and the distributions on the model parameters are supposed to push forward to these.

The main contribution of this paper is therefore to show how the minimum information approach can be used to generate solutions to the PI problem in the context of coupled models. The rest of the paper is structured as follows. In Section 2 we provide the mathematical setup to the problem, considering coupled models and minimum information modelling. In Section 3 we outline the solution to the continuous PI problem for coupled models and in Section 4 we give the equivalent solution to the

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discretized problem. Section 5 investigates the specification of feasible constraints for the coupled PI problem and considers the effect of fixing the constraints. In Sections 6 and 7 we provide two examples; the first considering the failure rates of machines located in series and the second based on the dispersion and deposition problem considered by Kraan in [7]. Finally, in Section 8, we give some conclusions and a discussion of areas for further work.

2. Mathematical setup

2.1. Coupled models

We consider models M_i taking input parameters y_i and giving output parameters y_{i+1}. So the output of model 1 is the input for model 2. The models are deterministic models, but are assumed to have parameters (which may be vectors) that we denote x_i. An illustration of this situation is given in Fig. 1.

Some of the parameters may be directly measurable. Some we might be able to choose. However, often the model parameters are not known directly and we have to infer them, or infer uncertainty distributions over them. The inputs and outputs of the models are observable quantities in principle, and we may therefore be able to use expert judgement to assess distributions or expected values. In probabilistic inversion we try to find a distribution for the parameters that matches (when propagated through the model) the distributions specified by the experts for the observable quantities.

As noted above, the PI problem is typically either over- or under-constrained. Over-constraint happens because the quantities to be assessed are typically functionally related through the model, so it is easy for experts to provide information on observables that is mutually inconsistent (assuming the model is correct). Under-constraint happens because there is consistency but there is still not enough information to uniquely determine the distribution on the parameters. In our approach we use a minimum information property to solve the problem of under-constraint, and a sequential approach to expert elicitation to avoid the problem of over-specification.

2.2. Minimum information modelling

Suppose we have continuous random variable X, which could be multi-dimensional, and two densities, $f_1(\cdot)$ and $f_2(\cdot)$. Then the relative information of $f_1(\cdot)$ to $f_2(\cdot)$ is a measure of how similar the two distributions are. It is defined as

$$I(f_1|f_2) = \int f_1(x) \log\left(\frac{f_1(x)}{f_2(x)}\right) dx.$$

If $f_1(\cdot) = f_2(\cdot)$ then the relative information is zero. It then increases as the deviation between the two distributions becomes greater.

In the problems considered in this paper we wish to find the distribution, $f_1(\cdot)$, which has minimum information, with respect to the background distribution $f_2(\cdot)$, subject to some real valued

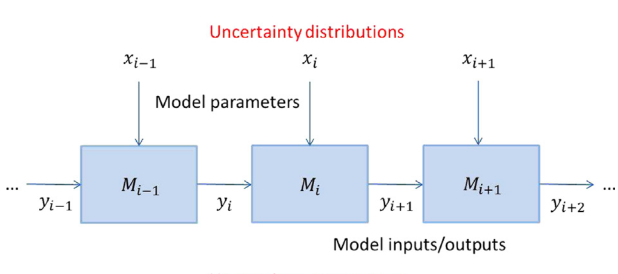

Fig. 1. A diagram illustrating the form of coupling of models.

functions $h_1 \ldots, h_k$ taking expectations $\alpha_1, \ldots, \alpha_k$. These are known as the *constraints*. The choice of background distribution is an important consideration as this specifies what $f_1(\cdot)$ should look like in the absence of more specific information. Typical choices might be the uniform, log-uniform and Normal distributions (but this can be subject to sensitivity analysis as we discuss later).

The minimum information distribution is therefore, in some sense, the "simplest" distribution which satisfies the required criteria. If, relative to the background distribution, this distribution exists then it is unique and takes the form [8]

$$f(x) = \frac{\exp\{\sum_{i=1}^{k} \lambda_i h_i(x)\}}{Z(\lambda)}, \tag{1}$$

for Lagrange multipliers $\lambda = (\lambda_1, \ldots, \lambda_k)$ (depending on $\alpha_1, \ldots, \alpha_k$), where $Z(\lambda)$ is the normalizing constant. An important contribution of this paper is a procedure which provides a method of specifying the constraints using expert elicitation which ensures such a unique minimally informative solution exists. We can approximate this density arbitrarily well using discrete densities

$$p(x_j) \propto \exp\left\{\sum_{i=1}^{k} \lambda_i h_i(x_j)\right\}, \tag{2}$$

where x_j, $j=1,\ldots,n$, is a suitable discretization of x. A derivation of these results is outlined in Bedford and Wilson [9].

We see that, for minimum information problems, constraints are naturally specified as expectations on functions of the parameters. We can incorporate quantile assessments as the expert judgement by specifying the h_i's as indicator functions in terms of the quantile functions of the parameters. An example is given in Bedford et al. [10] and the technique is used in the failure rates example in Section 6.

We shall now consider how to evaluate the feasible values of constraints when the specifications of those constraints are made sequentially. Initially we consider the continuous problem.

3. The continuous problem

Consider a collection of parameters from a deterministic model or sequence of models, denoted X. In general X could have a large number of dimensions. Suppose that expert elicitation results in the specification of the expectations of some functions, $h_1, \ldots, h_p$, of these parameters, denoted $\alpha_1, \ldots, \alpha_p$. These are the constraints in the problem. We wish to find the distribution with minimum information which satisfies these expectations.

As we saw in the previous section this minimum information distribution has a density at x which is proportional to

$$\exp\left\{\sum_{i=1}^{p} \lambda_i h_i(x)\right\},$$

for some parameters $\lambda_1 \ldots, \lambda_p$. We can define the function $\phi : \mathbb{R}^p \to \mathbb{R}^p$ so that ϕ gives the expected values obtained from choosing the minimum information distribution with parameters $\lambda_1, \ldots, \lambda_p$. That is,

$$\phi(\lambda_1 \ldots, \lambda_p) = (\alpha_1, \ldots, \alpha_p).$$

In particular,

$$\phi(\lambda_1, \ldots, \lambda_p) = \left(\frac{\int h_1(x) \exp\{\sum_i \lambda_i h_i(x)\}\, dx}{\int \exp\{\sum_i \lambda_i h_i(x)\}\, dx}, \ldots, \frac{\int h_p(x) \exp\{\sum_i \lambda_i h_i(x)\}\, dx}{\int \exp\{\sum_i \lambda_i h_i(x)\}\, dx}\right).$$

The function ϕ is invertible and has good analytical properties.

We wish to specify α_1 and use this to explore the possible values α_2 can take. Having done this the next step is to find the possible values for α_3 having specified α_1, α_2. We can continue in this way, evaluating the possible specifications of each expectation consistent with previously specified values, until all of the