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Analytical evaluation of the transverse displacement at the tip of a semi-infinite crack in an elastic plate

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ABSTRACT

In this paper, a general approach is developed for evaluating the transverse displacements in an angular sector. The problem is investigated within the first order plate theory, which can be considered as an elementary extension of the classical plane theories of elasticity. Based on this approach, a new analytical solution for a semi-infinite crack subjected to mode I loading is obtained and this solution is verified against previous three-dimensional Finite Element studies. This approach can also be useful in the analysis of other problems, which can be reduced to a modified Helmholtz (or Yukawa) equation in an angular sector.

1. Introduction

The simplest analytical theory, which is capable to evaluate the three-dimensional stress and displacement fields in plane problems of elasticity is the first order plate theory suggested by Kane and Mindlin in 1956. This theory was originally applied to the analysis of high-frequency extensional vibrations in moderately thick plates [1]. It is based on a kinematic assumption that the in-plane displacement components remain constant through the plate thickness and the transverse (out-of-plane) displacement component varies linearly across the plate thickness. For example, the displacement field in an elastic plate bounded by planes $z=\pm h$ is defined as [2]

$$u_r = u_r(r,\phi),\quad u_\phi = u_\phi(r,\phi),\quad u_z = \frac{z}{h}w(r,\phi), \tag{1}$$

where r and ϕ are the in-plane coordinates and z is the distance from the mid-plane (Fig. 1). Due to the underlying kinematic assumption, the first order plate theory cannot capture certain 3D effects, such as the 3D corner or vertex singularity, which dominates at distances of approximately 0.1 h from the vertex [17,18]. However, the kinematic assumption allows the theory to retain the simplicity of a 2D formulation for the analysis of 3D plane problems.

The stress resultants of the first order plate theory are defined as [2]:

$$\begin{aligned}\{N_{rr},N_{\phi\phi},N_{r\phi}\} &= \int_{-h}^{h}\{\sigma_{rr},\sigma_{\phi\phi},\sigma_{r\phi}\}dz, N_{zz}\\ &= \int_{-h}^{h}\sigma_{zz}dz, \{N_{rz},N_{\phi z}\}\\ &= \int_{-h}^{h} z\{\sigma_{rz},\sigma_{\phi z}\}dz.\end{aligned} \tag{2}$$

Due to the kinematic assumption (1), the in-plane stress resultants $\{N_{rr},N_{\phi\phi},N_{r\phi}\}$ are simply equal to the average through-the-thickness in-plane stress component, $\{\sigma_{rr},\sigma_{\phi\phi},\sigma_{r\phi}\}$, multiplied by the plate thickness, $2h$. The stress resultants, N_{rz} and $N_{\phi z}$ are the components of pinching shear. The latter play a role in extensional deformations similar to the transverse shearing forces in flexure.

If the mean in-plane stress resultant is denoted by $N=(N_{rr}+N_{\phi\phi})/2$, then the equilibrium equations of the first order plane theory result into the following relationship between the out-of-plane displacement function, $w(r,\phi)$ and the mean in-plane stress resultant, $N(r,\phi)$, see, for instance, Eq. (6a) in Yang and Freund [2]:

$$\frac{h^2}{3}\frac{\lambda+\mu}{3\lambda+2\mu}\nabla^2 w - w = \frac{1}{2\mu}\frac{\lambda}{3\lambda+2\mu}N. \tag{3}$$

Here ∇^2 is the Laplace operator, h is the half plate thickness, μ and λ are Lame constants. The governing equation (1) can also be rewritten in terms of Young's modulus, E and Poisson's ratio, ν as

$$\nabla^2 w - \kappa^2 w = \frac{\nu h^2\kappa^2}{E}N,\quad \kappa^2 = \frac{6}{h^2}(1+\nu). \tag{4}$$

Eq. (4) is coupled with three other governing equations of the first order plate theory: two equilibrium equations for the in-plane stress resultants, $N_{\alpha\beta,\beta}=0$, $\alpha,\beta=r,\phi$, and the strain compatibility equation, $\nabla^2(N-E\nu w/(1+\nu)^2)=0$. This coupling of four partial differential equations makes the analysis of non-trivial problems difficult.

In order to simplify the problem formulation, the present work assumes

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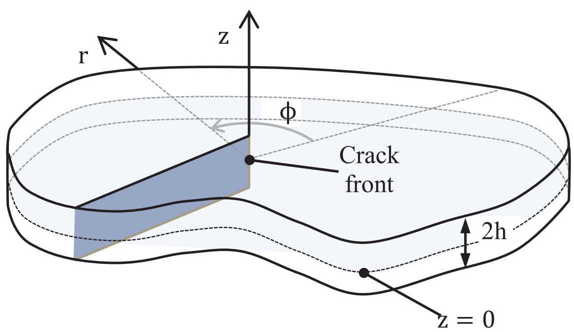

Fig. 1. Coordinate system and problem geometry.

$$\frac{(N_{rr}+N_{\phi\phi})}{2h}\approx\check{\sigma}_{rr}+\check{\sigma}_{\phi\phi},\tag{5}$$

where the stress components $\check{\sigma}_{rr}$ and $\check{\sigma}_{\phi\phi}$ are the known solution of the corresponding 2D plane stress problem. Indeed, analytical, numerical and experimental studies of plane problems of elasticity over the past hundred years arrived at two main conclusions: firstly, the classical plane stress solutions describe accurately the in-plane stress components of the actual three-dimensional stress state, and secondly, the variation of the in-plane stress components across the plate thickness is small and can be neglected for all practical purposes. These fundamental conclusions, in particular, justify the use of various solutions of the plane theory of elasticity for stress analysis and design of plate components across many industries and applications.

The simplified governing equation for $w(r,\phi)$, which is now uncoupled from the other governing equations of the first order plate theory, can be written as:

$$\nabla^2 w-\kappa^2 w=\frac{\nu h\kappa^2}{E}(\check{\sigma}_{rr}+\check{\sigma}_{\phi\phi}),\tag{6}$$

where the sum of in-plane normal stress components $(\check{\sigma}_{rr}+\check{\sigma}_{\phi\phi})$ is assumed to be a known function of the in-plane coordinates.

2. The approach

Because $\check{\sigma}_{11}$ and $\check{\sigma}_{22}$ represent the plane stress solution of the corresponding two-dimensional problem, for which

$$\nabla^2(\check{\sigma}_{rr}+\check{\sigma}_{\phi\phi})=\nabla^2(\nabla^2\Phi)=\nabla^4\Phi=0,\tag{7}$$

where Φ is the classical Airy stress function (bi-harmonic function), the solution of the governing equation (6) can be written as

$$w=w_p+w_h,\quad w_p=-\frac{\nu h}{E}(\check{\sigma}_{rr}+\check{\sigma}_{\phi\phi})\tag{8}$$

In (8), w_p represents the particular solution for the out-of-plane displacement corresponding to the plane stress solution. The function w_h corresponds to the solution of the homogeneous modified Helmholtz equation:

$$\nabla^2 w_h-\kappa^2 w_h=0.\tag{9}$$

One must find such a homogenous solution w_h, which ensures that:

1. the transverse displacement, $w(r,\phi)$, is finite everywhere, even at points where the particular solution, w_p, is singular (for e.g. at the crack tip), and
2. the traction-free boundary conditions must be satisfied along free edges (for e.g. crack faces).

In addition, the out-of-plane displacement function, w_h, has to decay with the distance from the edges, so the solution (8) converges to the plane stress solutions in the interior domain. The decaying solution of equation (9) describes a boundary layer effect [2] of characteristic length: $\kappa^{-1}\sim h$.

To demonstrate how the approach works, the problem of a semi-infinite crack in an elastic plate stressed in mode I and II is considered in more detail. From the plane stress solution of this problem, the sum of normal in-plane stresses is

$$\check{\sigma}_{rr}+\check{\sigma}_{\phi\phi}=\frac{2K_I}{\sqrt{2\pi r}}\cos\frac{\phi}{2}\quad\text{(Mode I)},$$
$$\check{\sigma}_{rr}+\check{\sigma}_{\phi\phi}=-\frac{2K_{II}}{\sqrt{2\pi r}}\sin\frac{\phi}{2}\quad\text{(Mode II)},\tag{10}$$

where the origin of the cylindrical polar coordinate system lies at the tip of the crack, $\phi=\pm\pi$ coincide with the crack faces and K_I and K_{II} are the remotely applied mode I and II stress intensity factors, respectively. The out-of-plane displacement corresponding to the plane stress solution is simply

$$w_p=-K_I\frac{\nu h}{E}\sqrt{\frac{2}{\pi r}}\cos\frac{\phi}{2}\quad\text{(Mode I)},$$
$$w_p=K_{II}\frac{\nu h}{E}\sqrt{\frac{2}{\pi r}}\sin\frac{\phi}{2}\quad\text{(Mode II)},\tag{11}$$

It is readily observed that the particular solutions for both modes are wholly analogous and are singular at $r=0$, with asymptotic behaviour of $r^{-1/2}$. To ensure the finiteness of the general solution, a homogenous solution, say w_{h1}, which has the same asymptotic behaviour as w_p close to $r=0$ but the opposite sign, must be obtained. One such solution admitted by the homogenous modified Helmholtz equation $\nabla^2 w-\kappa^2 w=0$ is

$$w_{h_1}=K_I\frac{\nu h}{E}\frac{2\sqrt{\kappa}}{\pi}K_{1/2}(\kappa r)\cos\frac{\phi}{2}\quad\text{(Mode I)},$$
$$w_{h_1}=-K_{II}\frac{\nu h}{E}\frac{2\sqrt{\kappa}}{\pi}K_{1/2}(\kappa r)\sin\frac{\phi}{2}\quad\text{(Mode II)},\tag{12}$$

where $K_{1/2}(\kappa r)$ is the modified Bessel function of second kind. The above expression can also be re-written as

$$w_{h_1}=K_I\frac{\nu h}{E}\sqrt{\frac{2}{\pi r}}e^{-\kappa r}\cos\frac{\phi}{2}\quad\text{(Mode I)},$$
$$w_{h_1}=-K_{II}\frac{\nu h}{E}\sqrt{\frac{2}{\pi r}}e^{-\kappa r}\sin\frac{\phi}{2}\quad\text{(Mode II)},\tag{13}$$

Adding the two solutions yields an intermediate solution to the non-homogenous differential equation (6), $w_*=w_p+w_{h_1}$, such that

$$w_*=K_I\frac{\nu h}{E}\sqrt{\frac{2}{\pi r}}(e^{-\kappa r}-1)\cos\frac{\phi}{2}\quad\text{(Mode I)},$$
$$w_*=-K_{II}\frac{\nu h}{E}\sqrt{\frac{2}{\pi r}}(e^{-\kappa r}-1)\sin\frac{\phi}{2}\quad\text{(Mode II)},\tag{14}$$

These solution, as expected, converges to the plane stress solution at the distance $\kappa\, r>1$ from the crack tip.

It must now be verified whether the intermediate solution w_* satisfies the traction free boundary condition along the crack faces, i.e. along $\phi=\pm\pi$. The three traction components acting on these faces are: $N_{\phi\phi}$, $N_{r\phi}$ and $N_{\phi z}$. The in-plane stress resultants satisfy this requirement. The requirement of $N_{\phi z}=0$ can be written as $\partial w/\partial\phi=0$. For the two modes of loading, the latter derivative can be found as

$$\left.\frac{\partial w_*}{\partial\phi}\right|_{\phi=\pm\pi}=-\frac{\nu h}{E}\frac{K_I}{\sqrt{2\pi r}}(e^{-\kappa r}-1)\quad\text{(Mode I)},\qquad\left.\frac{\partial w_*}{\partial\phi}\right|_{\phi=\pm\pi}=0\quad\text{(Mode II)}.\tag{15}$$

It becomes apparent that the intermediate solution w_* only satisfies the traction-free boundary condition at the free edges of the plate i.e. $\phi=\pm\pi$ for the case of Mode II loading.

The obtained solution for Mode II loading agrees very well with both experimental observations, [3,4] and outcomes of careful 3D Finite Element studies e.g. [5–8], with more details found in [9]. Therefore, the derived governing equation agrees well with works of