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Letter

An analysis of dynamic stability of an elastic column

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ABSTRACT

By the Lyapunov direct method, dynamic stability of two conservative systems of finite degrees of freedom with one parameter is analyzed. Two Lyapunov functions are proposed for the two systems, respectively. When the number of degree of freedom the two systems tends to infinite, the two systems can simulate dynamic stability of a compressed elastic column with one end fixed and the other clamped in rotation. In the sense of the Lyapunov stability, the column is proved to be dynamically stable when the load equals to the Euler critical load.

1 Introduction

Dynamic stability of a compressed elastic column is a key problem in structure analysis. The problem has received considerable attention in early years, for examples, Woinowsky-Krieger [1], Brown et al. [2], Dickey [3], Reiss and Matkowsky [4], Tseng and Dugundji [5], Ball [6] and others. In recent years, Abou-Rayan et al. [7], Afaneh and Ibrahim [8], Nayfeh et al. [9], Chin and Nayfeh [10], Kreider and Nayfeh [11], Nayfeh and Emam [12-14], Mamandi et al. [15], Yang and Zhang [16], Emam and Abdalla [17], and Ghayesh and Farokhi [18] have been devoted to analyzing the non-linear response of buckled beams, which is still open for research. A basic problem is the dynamic stability of a column in straight shape in compression. Movchan [19] proved that a column is dynamically stable when compression load is smaller than the Euler load. However, in the sense of the Lyapunov stability, it is not clear theoretically up till now whether or not a column just at the first bifurcation point is dynamically stable and unstable when the load is greater than the Euler critical load. Not only the construction and analysis of the Lyapunov functional is the key problem in the dynamic stability of structures, but also the key problem in the other areas, such as control theory [20-22]. In this paper, by the Lyapunov functionals proposed by the author, the dynamic stability of two conservative systems is analyzed. The dynamic stability of a column with one end fixed and the other clamped in rotation is proved theoretically.

The paper is organized as follows: Section 2 is some basics, including definition in the sense of the Lyapunov stability, the Lyapunov theorems on dynamical stability and total energy of the column in vibration. Section 3 presents a conservative system of finite degree of freedom with a parameter, i.e. system I, an analysis of the dynamic stability of system I and a column with one end fixed and the other clamped in rotation when the load is smaller than or equals to the Euler critical load. Section 4 presents another conservative system of finite degree of freedom with a parameter, i.e. system II, an analysis of the dynamic instability of system II and the column when the load is greater the Euler critical load. Section 5 is conclusions.

2 Some basics

2.1 Lyapunov direct method

Consider a l-degree-of-freedom dynamical system, the state variable is denoted by

$$\boldsymbol{\varepsilon}_l = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_l; \dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\varepsilon}_3, \cdots, \dot{\varepsilon}_l)^{\mathrm{T}}, \quad l = 1, 2, 3, \cdots, \tag{1}$$

where ε_i and $\dot{\varepsilon}_i$ $(i = 1, 2, 3, \cdots, l)$ are the general displacements and the general velocities, respectively. Vector $\boldsymbol{\varepsilon}_l = 0$ is the null

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solution or an equilibrium point of the l-degree-of-freedom system. The Euclidean norm of ε_l is

$$\|\varepsilon_l\| = \sqrt{\sum_{i=1}^{l}\left(\varepsilon_i^2 + \dot{\varepsilon}_i^2\right)}. \tag{2}$$

In the sense of Lyapunov, stable and unstable of the null solution $\varepsilon_l = 0$ on dynamic stability are defined as [23]:

1. The null solution is stable in the sense of Lyapunov if for any arbitrary positive γ and time t_0 , there exists a $\eta = \eta(\gamma, t_0) > 0$ such that if the inequality

$$\|\varepsilon_l(t_0)\| < \eta, \tag{3}$$

is satisfied, then the inequality

$$\|\varepsilon_l(t)\| < \gamma,\ t \in [t_0, \infty), \tag{4}$$

is implied.

2. The null solution is said to be unstable in the sense of Lyapunov if for any arbitrary small η and any time t_0 such that

$$\|\varepsilon_l(t_0)\| < \eta. \tag{5}$$

We have at some other finite time t_1 the situation

$$\|\varepsilon_l(t_1)\| = \gamma,\ t_1 > t_0, \tag{6}$$

where γ is a given arbitrary positive number. There are two sufficient conditions on stability and instability, respectively [23].

The Lyapunov stability theorem: If there exists a positive-definite function $V(\varepsilon_l)$ whose total time derivative $\dot{V}(\varepsilon_l)$ is negative semi-definite along every trajectory of the system, then the trivial solution $\varepsilon_l = 0$ is stable. $V(\varepsilon_l)$ is called the Lyapunov function, and $\dot{V}(\varepsilon_l) = \frac{\mathrm{d}V}{\mathrm{d}t}$ is the total time derivative of $V(\varepsilon_l)$ along every trajectory of the system, where t is the time.

The Lyapunov instability theorem: If there exists for the system a function $U(\varepsilon_l)$ whose total time derivative $\dot{U}(\varepsilon_l)$ is positive-definite along every trajectory of the system and the function itself can assume positive values for an arbitrarily small values of ε_l, then the trivial solution $\varepsilon_l = 0$ is unstable. $U(\varepsilon_l)$ is also called the Lyapunov function of the system.

2.2 Total energy of a column in vibration

An inextensible planar column with one end fixed and the other clamped in rotation is shown in Fig. 1. EI denotes the bending rigidity of the column, b denotes the length of the column before deformation, N denotes an axial compression load, w and θ denote lateral deflection and the tangential angle of the lateral deflection curve, respectively. $s \in [0, b]$ and $h \in [0, s]$ denote two arc-length coordinates with origin o on the axial line before deformation.

The total energy of the system is sum of potential energy and kinetic energy. Elastic potential energy equals to the strain energy. Dimensionless form of the strain energy is

$$\int_0^1 \frac{1}{2}\theta'^2 \mathrm{d}x, \tag{7}$$

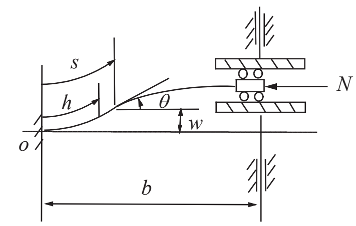

Fig. 1. An inextensible planar column with one clamped-end and the other clamped in rotation

where $x = \frac{s}{b}$ and $\theta' = \frac{\partial\theta}{\partial x}$. Dimensionless form of potential energy of the external force N is

$$k^2 \int_0^1 (\cos\theta - 1)\,\mathrm{d}x, \tag{8}$$

where $k = b\sqrt{\frac{N}{EI}}$ is the load factor. Dimensionless form of translational kinetic energy is

$$\int_0^1 \frac{1}{2}\alpha\left(\int_0^x \dot{\theta}\cos\theta\,\mathrm{d}\xi\right)^2 \mathrm{d}x, \tag{9}$$

where $\alpha = \frac{mb^4}{EI}$, m is the mass per unit length of the column, $\dot{\theta} = \frac{\partial\theta}{\partial t}$ is the angular velocity, and $\xi = \frac{h}{b}$. Dimensionless form of rotational kinetic energy is

$$\int_0^1 \frac{1}{2}\beta\dot{\theta}^2 \mathrm{d}x, \tag{10}$$

where β is the moment of inertia per unit length of the column. Summing up Eqs. (7)-(10), we have dimensionless form of the total energy of the system

$$V = \int_0^1 \left[\frac{1}{2}\theta'^2 + k^2(\cos\theta - 1) + \frac{1}{2}\alpha\left(\int_0^x \dot{\theta}\cos\theta\,\mathrm{d}\xi\right)^2 + \frac{1}{2}\beta\dot{\theta}^2\right]\mathrm{d}x \tag{11}$$

Boundary conditions on the tangent angle of the column in Fig. 1 are

$$\theta(0, t) = 0,\ \theta(1, t) = 0. \tag{12}$$

The load factor at the first bifurcation point is $k = \pi$. Considering conditions in Eq. (12), we have the Fourier series of the tangential angle

$$\theta(x, t) = \sum_{i=1}^{\infty} \varepsilon_i(t)\sin(i\pi x),\ \ |\theta| < 1. \tag{13}$$

Based on the Hamilton principle, trajectory of the system makes functional