Theoretical & Applied Mechanics Letters 7 (2017) 292–300

Contents lists available at ScienceDirect

Theoretical & Applied Mechanics Letters

journal homepage: www.elsevier.com/locate/taml

Letter

Multiple solutions and stability of the steady transonic small-disturbance equation

Ya Liu [a], Shijun Luo [b], Feng Liu [b,*]

[a] College of Engineering, Peking University, Beijing 100871, China
[b] Department of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92697, USA

A R T I C L E I N F O

Article history:
Received 14 September 2017
Accepted 15 September 2017
Available online 25 October 2017
*This article belongs to the Fluid Mechanics.

Keywords:
Transonic flow
Multiple solutions
Flow stability
Numerical method
Potential flow

A B S T R A C T

Numerical solutions of the steady transonic small-disturbance (TSD) potential equation are computed using the conservative Murman—Cole scheme. Multiple solutions are discovered and mapped out for the Mach number range at zero angle of attack and the angle of attack range at Mach number 0.85 for the NACA 0012 airfoil. We present a linear stability analysis method by directly assembling and evaluating the Jacobian matrix of the nonlinear finite-difference equation of the TSD equation. The stability of all the discovered multiple solutions are then determined by the proposed eigen analysis. The relation of stability to convergence of the iterative method for solving the TSD equation is discussed. Computations and the stability analysis demonstrate the possibility of eliminating the multiple solutions and stabilizing the remaining unique solution by adding a sufficiently long splitter plate downstream the airfoil trailing edge. Finally, instability of the solution of the TSD equation is shown to be closely connected to the onset of transonic buffet by comparing with experimental data.

The non-uniqueness of numerical solutions of potential equations at transonic speeds has been found for three decades. Steinhoff and Jameson [1] first reported multiple solutions for the full potential (FP) equation. Chen [2] first reported the existence of multiple solutions of the steady transonic small-disturbance (TSD) equation using the nonconservative Murman—Cole scheme. Nixon [3] also found multiple solutions using the TSD equation modified with vorticity and entropy corrections. Salas et al. [4,5] did extensive study on multiple solutions of the FP equation. Jameson [6] demonstrated that non-unique solutions of Euler equations can be obtained for certain airfoils. Hafez and Guo [7,8] investigated the flow over airfoils with flat and wavy surface by solving the steady potential equations, the Euler equations, and the Navier—Stokes equations. They found that all of the equations can generate multiple solutions at zero angle of attack in certain Mach number ranges. Luo et al. [9] showed that the multiple solutions of the transonic small transverse disturbance equation are independent of the difference schemes and iterative methods and found multiple solutions for a three dimensional wing.

Williams et al. [10] investigated the stability of multiple solutions of the unsteady TSD equation by a time-marching method and showed that the asymmetric solution results from an extremely long time scale instability of the symmetric solution. Caughey [11] investigated the stability of nonunique solutions of the steady Euler equations by time-accurate simulations of the unsteady flow past airfoils for which there exhibit nonunique solutions. Bailey and Beam [12] evaluated the temporal stability of the steady-state solutions of the compressible Navier—Stokes equations in two dimensions by freezing the Jacobian matrices (approximate Newton's method) of the finite-difference approximations.

In a recent paper, Kuzmin [13] presented a comprehensive review of multiple-solution phenomenon for transonic flow over different airfoils. Multiple solutions of the Euler and Reynolds averaged Navier—Stokes (RANS) equations are discussed in detail. The airfoils admitting multiple solutions generally have a long flat segment, and the instability of the solutions is attributed to the rupture/coalescence of supersonic regions.

The present paper is undertaken to study the stability of multiple solutions of the steady TSD equation with the eigenvalue technique. Numerical solutions of the steady TSD equation are first obtained using the conservative Murman—Cole scheme and are considered as equilibrium points. Then the Jacobian matrix is constructed from the discrete steady TSD equations and evaluated using the numerical solution. The eigenvalues of the Jacobian matrix are calculated and the stability of the numerical solution is

* Corresponding author.
E-mail address: fliu@uci.edu (F. Liu).

https://doi.org/10.1016/j.taml.2017.09.011

Nomenclature

A	Jacobian matrix
C_l	lift coefficient
c	chord length
I, J	rectilinear grid number
i, j	rectilinear grid index
i_l	grid index at the leading edge of the airfoil
i_t	grid index at the trailing edge of the airfoil
M_∞	freestream Mach number
$\boldsymbol{q}$	solution vector
q_∞	freestream velocity
Re	Reynolds number
x, y	Cartesian coordinates
α	angle of attack
γ	ratio of specific heats
λ	eigenvalues of the Jacobian matrix
ν	number of iterations
ϕ	perturbation velocity potential
ε	maximum residual

identified. The relation between stability of the numerical solution and convergence of the iterative method is discussed. Then the elimination of the multiple solutions by attaching a splitter plate to the airfoil trailing edge is studied, its physical significance is discussed. Finally, comparisons with experimental transonic buffet data infer that the instability of the TSD solution may be closely related to the onset of buffet.

The governing equation is the two-dimensional steady TSD equation

$$(1 - M_\infty^2 - \frac{\gamma + 1}{q_\infty} M_\infty^2 \phi_x)\phi_{xx} + \phi_{yy} = 0, \tag{1}$$

where M_∞ and q_∞ are the freestream Mach number and velocity, respectively, γ is the ratio of specific heats, ϕ is the perturbation velocity potential, and the x-axis is parallel to the airfoil chord.

The conservative Murman–Cole difference scheme [14] is used to write Eq. (1) into a difference equation over a rectilinear grid (i, j). The grid line j = const. is parallel to the x-axis. The airfoil is located on the x-axis where $j = jw$. The airfoil chord has unit length. The leading and trailing edge of the airfoil is located at $x = -0.5$ or $i = i_l$ and $x = 0.5$ or $i = i_t$, respectively. Starting from the airfoil leading edge till downstream far field, the grid line $j = jw$ is split into two grid lines $j = jw + 0$ and $j = jw - 0$, which denote the upper and lower surface of the airfoil chord and wake, respectively. The far field boundaries are located on the grid lines, they are $i = 1$ and I, and $j = 1$ and J.

To compute the transonic flow about an NACA 0012 airfoil at small angle of attack, a grid $I = 81$ and $J = 41$ is used. $i_l = 21$, $i_t = 61$, 40 uniform grids in x-direction are used over the airfoil. Away from the airfoil, one-dimensional stretch function is used to generate the non-uniform grids. The far field extends to 40 times chord length of the airfoil. The grid is set to be symmetric with respect to both x- and y-axes. The local grids around the leading edge of the airfoil are shown in Fig. 1.

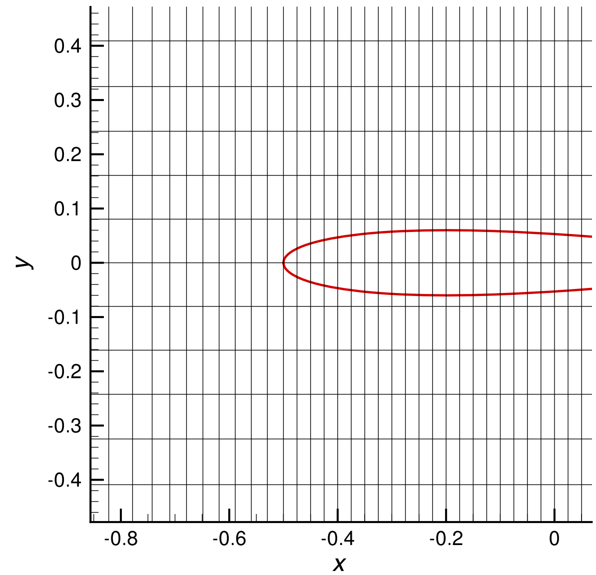

Fig. 1. Local view of grids around airfoil leading part.

The boundary conditions on the airfoil upper and lower surfaces are applied on the grid line $j = jw + 0$ and $j = jw - 0$, respectively.

$$(\phi_y)_{i,jw+0} = q_\infty(\frac{\mathrm{d}y_u}{\mathrm{d}x} - \alpha), \tag{2}$$

$$(\phi_y)_{i,jw-0} = q_\infty(\frac{\mathrm{d}y_l}{\mathrm{d}x} - \alpha), \tag{3}$$

where y_u and y_l are the y-coordinate of the airfoil upper and lower surface, respectively, and α is the angle of attack. At the grid point of the blunt leading edge, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is empirically replaced by that at the grid point immediately behind the leading edge. The Kutta conditions are also applied on the x-axis.

$$(\phi_y)_{i,jw+0} = (\phi_y)_{i,jw-0} = (\phi_y)_{i,jw}, \tag{4}$$

$$\phi_{i,jw+0} - \phi_{i,jw-0} = \phi_{i_t,jw+0} - \phi_{i_t,jw-0}. \tag{5}$$

The airfoil boundary conditions and the wake Kutta conditions are embedded into the difference equations at appropriate grid points. The Kutta condition Eq. (4) introduces a new unknown variable $(\phi_y)_{i,jw}$ besides the discrete $\phi_{i,j}$. The Kutta condition Eq. (5) provides the required additional equation besides the difference equations. The far field boundary condition is approximated by $\phi = 0$ for simplifying the computation and is verified in the following. The number of algebraic equations is exactly equal to the total number of unknowns. The system of nonlinear algebraic equations is solved using the successive line over relaxation (SLOR) method.

The multiple solutions of the TSD equation for the NACA 0012 airfoil at zero angle of attack and transonic speeds are computed. The symmetric solution is obtained using the freestream flow as the initial condition. The corresponding asymmetric solution, if exists, is found using the solution for the NACA 0012 airfoil at $\alpha = 1°$ and the same M_∞ as the initial condition. The mirror image of such a solution is another asymmetric solution. Double precision is used in the computation. When the maximum residual of the computation drops from 10^{-2} to 10^{-10}, the numerical solution is considered as being convergent.

Figure 2 presents the pressure distributions of the symmetric and asymmetric solutions obtained at $M_\infty = 0.84$ and $\alpha = 0$, compared with those obtained by Williams et al. [10], who solved the time-accurate TSD equation using the conservative Engquist–Osher difference scheme [15]. The two computational results agree well except near the shock waves. Similar multiple solutions at zero angle of attack for the NACA 0012 airfoil are found at $M_\infty = 0.85$ and 0.86 by the above method. The symmetric and asymmetric pressure distributions for $M_\infty = 0.85$ and 0.86 are shown in Fig. 3 and Fig. 4, respectively. On the other hand, the solutions for $M_\infty = 0.82, 0.83, 0.87$ are all unique and symmetric. An increment of 0.01 of the freestream Mach number is used in the present study in the search for the multiple solutions.