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Letter

A minimum-domain impulse theory for unsteady aerodynamic force with discrete wake

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ABSTRACT

We extend the impulse theory for unsteady aerodynamics, from its classic global form to finite-domain formulation, then to a *minimum-domain* version for discrete wake. Each extension has been confirmed numerically. The minimum-domain theory indicates that the numerical finding of Li and Lu (2012) is of general significance: The entire force is completely determined by only the time rate of impulse of those vortical structures still connecting to the body, along with the Lamb-vector integral thereof that captures the contribution of all the rest disconnected vortical structures.

The impulse theory pioneered by Burgers [1] and later developed independently by Wu [2, 3] and Lighthill [4], among others, serves as a primary theoretical tool in unsteady aerodynamics, in particular in the field of biological locomotion. For example, among others, by using the impulse theory, Hamdani and Sun [5] found that during the impulsive starts of a two-dimensional wing, the large vortex at trailing-edge during fast pitching-up rotation causes a large aerodynamic force; Birch and Dickinson [6] examined the influence of wing-wake interactions on the production of aerodynamic forces in flapping flight; Wang and Wu [7] identified the roles of vortex rings in lift production or reduction; Kim et al. [8] investigated vortex formation and force generation of clapping plates with various aspect ratios and stroke angles; and most recently, Andersen et al. [9] studied the close relation between the wake patterns and transition from drag to thrust on a flapping foil.

Despite its great generality and neatness, however, the classic form of impulse theory has an inherent limitation. It requires calculating the entire vorticity field in externally unbounded domain, or we may say that the theory is of global form. In contrast, the domain in computational and experimental fluid dynamics (CFD and EFD) is always bounded by a finite control surface Σ, say, with some vorticity inevitably going out of Σ shortly after the body motion starts. Hence, so far the impulse theory has been mostly confined to dealing with a sudden-start motion or flapping wings before the body-generated vorticity escapes out of a finite domain, beyond which the theory is invalid. Evidently, it is highly desired to extend the theory to a finite-domain form, such that for any body motion and deformation one can always utilize the vorticity-distribution data provided by CFD/EFD to diagnose the force constituents.

Mathematically, a general impulse formulation of aerodynamic force for arbitrary finite domain has been given by Noca et al. [10, 11]. Although the force prediction by the general formulation is accurate, however, it contains cumbersome boundary-integral terms with complicated physical meaning, making it difficult to pinpoint the dominant dynamic mechanisms responsible for the force. So the issue is whether the formulation can be significantly simplified to a powerful theoretical–physical tool for practical applications.

Here, a key physical observation is: unlike steady flows where the wake is always continuous, unsteady wakes behind flapping wings are often (though not always) discrete, for which the finite-domain impulse theory can be greatly simplified, which can then clearly reveal some simple physics of crucial importance. A pioneering work in this direction was made by Li and Lu [12] in a theoretical–numerical study of viscous and unsteady wake generated by flapping plates in relatively slow forward motion. There, the wake was found to be two rows of almost discrete vortex rings. The authors presented a finite-domain impulse formulation, and then found numerically that, the force of flapping plate is dominated by just the two vortices that still connect to the body.

This Letter proves theoretically, and confirms numerically, that the finding of Ref. [13] is of general significance: As long as wake vortices are discrete, the analysis domain in CFD/EFD can be minimized to a zone enclosing the body and those body-connected vortical structures. The force is solely determined by the time rate of the impulse of body-connected structures and a Lamb-vector integral thereof. The latter captures the contribution of all disconnected vortices in the wake.

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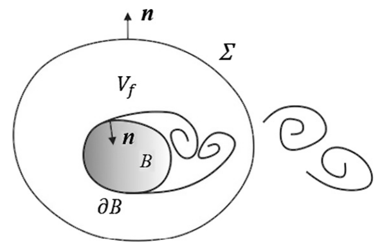

Fig. 1. Definition of notions for a moving/deforming body through the fluid in an arbitrary control volume $V = V_f + B$ bounded by Σ. The boundary of fluid V_f alone is $\partial V_f = \Sigma + \partial B$.

Consider incompressible flow with density $\rho = 1$ for neatness. For an arbitrary moving/deforming body, using the notation shown in Fig. 1, the total force is Let ∂B and Σ be material surfaces, and for convenience we set the density $\rho = 1$. Then the total force acting on the body is

$$\boldsymbol{F} = -\int_{\partial B}(-p\boldsymbol{n} + \boldsymbol{\tau})\mathrm{d}S \tag{1a}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}_f}\boldsymbol{u}\mathrm{d}V + \int_{\Sigma}(-p\boldsymbol{n} + \boldsymbol{\tau})\mathrm{d}S, \tag{1b}$$

where $\boldsymbol{\tau} = \nu\boldsymbol{\omega} \times \boldsymbol{n}$ is the shear stress with $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ being the vorticity. The central issue of the impulse theory is to express Eq. (1b) by the time rate of impulse to bypass the non-compact part of the total momentum that has poor divergence as $|\boldsymbol{x}| \to \infty$[13,14]. Some algebraic details are outlined in Appendix. By Eq. (A.1a), we split the total momentum in V_f into two parts:

$$\boldsymbol{P}_f \equiv \int_{V_f}\boldsymbol{u}\mathrm{d}V = \boldsymbol{I}_f - \boldsymbol{S}_f, \tag{2}$$

where

$$\boldsymbol{I}_f \equiv \frac{1}{k}\int_{V_f}\boldsymbol{x} \times \omega\mathrm{d}V, \tag{3a}$$

$$\boldsymbol{S}_f \equiv \frac{1}{k}\int_{\partial V_f}\boldsymbol{x} \times (\boldsymbol{n} \times \boldsymbol{u})\mathrm{d}S = \boldsymbol{S}_{\Sigma} + \boldsymbol{S}_B. \tag{3b}$$

Here $\boldsymbol{I}_f$ is the vortical impulse of V_f with $k = n - 1$, $n = 2, 3$ being the spatial dimension, the subscripts Σ and B denote surface integrals over external and internal boundaries of V_f, respectively. Then Eq. (1b) yields

$$\boldsymbol{F} = -\frac{\mathrm{d}\boldsymbol{I}_f}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{S}_B}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{S}_{\Sigma}}{\mathrm{d}t} + \int_{\Sigma}(-p\boldsymbol{n} + \boldsymbol{\tau})\mathrm{d}S. \tag{4}$$

For latter use, the kinematic content of $\mathrm{d}\boldsymbol{S}_f/\mathrm{d}t$ and the dynamic content of $\mathrm{d}\boldsymbol{I}_f/\mathrm{d}t$ are given in Appendix. Owing to the physical compactness of vorticity field, $\boldsymbol{I}_f$ surely remains finite. In contrast, $\boldsymbol{S}_{\Sigma}$ represents a non-compact part of $\boldsymbol{P}_f$. We have proved rigorously (not shown here) that if the domain $V = V_f + B$, which may well be finite, contains the entire compact vorticity field, then the third and fourth terms of Eq. (4) are canceled and the global form follows.

For a material Σ, substituting Eq. (A.5) into Eq. (4), we obtain the impulse formulation in arbitrary material finite domain

$$\boldsymbol{F} = -\frac{\mathrm{d}\boldsymbol{I}_f}{\mathrm{d}t} - \int_{\mathcal{V}_f}\boldsymbol{\omega} \times \boldsymbol{u}\mathrm{d}V + \boldsymbol{F}_{\partial B} + \frac{1}{k}\int_{\Sigma}\boldsymbol{x} \times \boldsymbol{u}\omega_n\mathrm{d}S + \boldsymbol{F}_{\Sigma}, \tag{5}$$

where

$$\boldsymbol{F}_{\partial B} \equiv \frac{1}{k}\int_{\partial B}\boldsymbol{x} \times (\boldsymbol{n} \times \boldsymbol{a})\mathrm{d}S + \frac{1}{k}\int_{\partial B}\boldsymbol{x} \times \boldsymbol{u}\omega_n\mathrm{d}S, \tag{6}$$

and

$$\boldsymbol{F}_{\Sigma} \equiv \frac{1}{k}\int_{\Sigma}(\boldsymbol{x} \times \boldsymbol{\sigma} + \boldsymbol{\tau})\mathrm{d}S, \tag{7}$$

is a viscous effect at Σ with $\boldsymbol{\sigma} = \nu\partial\boldsymbol{\omega}/\partial n$ being the vorticity diffusive flux.

The Lamb-vector term is the effect of the vortical flow outside $\mathcal{V}_f$ (including the virtual fluid in B) on the force [15]. Physically, one may replace $\boldsymbol{\omega} \times \boldsymbol{u}$ in this term by $\boldsymbol{\omega} \times \nabla\phi_e$, where $\nabla\phi_e$ is the potential velocity induced by vorticity outside $\mathcal{V}_f$ – although this is inconvenient for CFD/EFD – and hence when all the rest vorticity outside Σ is sufficiently far from $\mathcal{V}_f$ then this vortex-force term may be neglected.

All boundary-integral terms in Eq. (5) have neater and clearer physical meaning than that of Noca et al. [10, 11]. $\boldsymbol{F}_{\partial B}$ represents the explicit effect of body motion and deformation, which for active motion/deformation is a prescribed integral due to the adherence of $\boldsymbol{u}$, $\boldsymbol{a}$, and ω_n at ∂B, and is completely independent of the flow field. Thus, $\boldsymbol{F}_{\partial B}$ serves as a driving mechanism of the flow field, which is in contrast to the rest terms of Eq. (5) that represent the fluid reaction to the body's driving.

Equation (5) is exact and general for incompressible flow, and fully equivalent to the unsteady vortex-force theory by using the Reynolds transport theorem [16]. Thus it is applicable to both steady and unsteady flow. But our concern here is only unsteady flow. Unlike $\boldsymbol{F}_{\partial B}$ which is inevitable for moving-deforming body, the Σ-integrals in Eq. (5) are no more than a necessary artifice to express the force of the body in externally unbounded fluid by the flow data in finite domain. They just reflect the effects of flow outside V on that inside, but their appearance makes the characteristic neatness of impulse theory completely lost, so the key physical mechanism for producing force is covered up. A natural step to simplify Eq. (5) is evidently to remove the Σ-integrals.

As exemplified by huge amount of observations including [13], in many cases the wake consists of discrete or compact vortical structures. This fact has made it possible to approximate the wake vortex street by point vortices as von Kármán [17] did and later by arrays of vortex patches [15]. Although in real unsteady flow the wake vortices may still be connected by vortex sheets, compared to concentrated vortices formed by vortex sheets, before rolling into tight concentrate vortices the vorticity therein per unit length is much weaker, and the intersections of Σ and the sheets are too small to have appreciable effect on the force. Therefore, we can choose special Σ to avoid cutting any discrete concentrate vortices by requiring

$$\boldsymbol{\omega} = \boldsymbol{0} \quad \text{at and near } \Sigma. \tag{8}$$

We call an outer boundary satisfying Eq. (8) a good Σ and a fluid domain bounded by a good Σ a good $\mathcal{V}_f$. By Eq. (8) all surface integrals over Σ in Eq. (5) disappear.

There can be more than one good Σ's when the wake has many discrete vortical structures. But for any compact vortical domain, say $\mathcal{V}_{f\mathrm{wk}}$, bounded by a good Σ, by Eqs. (A.7) and (8) the fluid exerts no force to the body apparently:

$$\boldsymbol{F}_{f\mathrm{wk}} = -\frac{\mathrm{d}\boldsymbol{I}_{f\mathrm{wk}}}{\mathrm{d}t} - \int_{\mathcal{V}_{f\mathrm{wk}}}\boldsymbol{\omega} \times \boldsymbol{u}\mathrm{d}V = \boldsymbol{0}, \tag{9}$$

no matter how complex the vortical structures could be in $\mathcal{V}_{f\mathrm{wk}}$. The same conclusion was made by Saffman [15] for inviscid flow with $\partial\mathcal{V}_f$ satisfying $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$. In fact, Eq. (9) implies that $\mathrm{d}\boldsymbol{I}_{f\mathrm{wk}}/\mathrm{d}t$ equals to the Lamb-vector integral over $V_\infty - \mathcal{V}_{f\mathrm{wk}}$.

This being the case, of good $\mathcal{V}_f$'s we can always identify a "body-connected zone" $\mathcal{V}_{f\mathrm{con}}$ bounded by a good Σ, like that sketched in Fig. 1, which contains all vortical structures still connecting to the body, including attached boundary layers, separated shear layers and rolled-up vortices thereby. Then the rest of $\mathcal{V}_f$ will certainly belong to $\mathcal{V}_{f\mathrm{wk}}$ and has no net contribution to Eq. (5). Therefore, we obtain the desired minimum-domain force formula by impulse:

$$\boldsymbol{F} = -\frac{\mathrm{d}\boldsymbol{I}_{f\mathrm{con}}}{\mathrm{d}t} - \int_{\mathcal{V}_{f\mathrm{con}}}\boldsymbol{\omega} \times \boldsymbol{u}\mathrm{d}V + \boldsymbol{F}_{\partial B}, \tag{10}$$