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Scalar-valued measures of stress dispersion

Ke Gao*, John P. Harrison

Department of Civil Engineering, University of Toronto, Toronto, Canada M5S 1A4

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ABSTRACT

In situ stress is an important parameter in rock mechanics, but localised measurements often display significant variability; for meaningful analyses it is essential that such variability is appropriately quantified. Among many statistics, dispersion, which denotes how scattered or spread out a data group is, is an effective tool to quantify the amount of variability. However, dispersion measures are commonly only used for scalar and vector data, and it is not yet clear what robust scalar-valued measures of stress dispersion – i.e. measures that are faithful to the tensorial nature of stress – are available. Here, using stress tensors referred to a common Cartesian coordinate system, we consider several dispersion measures, namely, Euclidean dispersion (a tensor version of standard deviation), and the three widely used multivariate dispersions of total variation, generalised variance and effective variance, for scalar-valued quantification of stress variability and to improve the existing related work. We compare these measures, show how they are linked to the covariance matrix of tensor components, and derive their invariance with respect to change of coordinate system. Through the use of synthetic two-dimensional stress data we demonstrate that these measures can effectively characterise the dispersion of stress data. Further analysis of randomly generated three-dimensional stress data reveals that generalised variance and effective variance, which consider both variances of, and covariances between, tensor components, are more effective than Euclidean dispersion and total variation which ignore covariances. The transformational invariance of generalised variance and effective variance allows these measures to be applied in any convenient coordinate system.

1. Introduction

In situ stress is an important parameter for a wide range of endeavours in rock mechanics, including rock engineering design, hydraulic fracturing analysis, rock mass permeability and evaluation of earthquake potential.[1–5] The stress in rock often displays significant variability,[4,6–9] and as an example Fig. 1 shows the dramatic change in terms of both principal stress magnitude and orientation that can be observed in a small zone.[7] The stress variability may be influenced by various factors such as intrinsic variation caused by the inherent variability of discontinuities, anisotropy and heterogeneity of a fractured rock mass, as well as extrinsic errors related to stress acquisition methods. The acquisition error can be attributed to many aspects such as poor instrument installation, inaccurate estimation of mechanical parameters which are used in stress calculation, precision of the acquisition instruments, as well as the assumptions and constraints that have been made regarding the principal stresses in methods like hydraulic fracturing and borehole breakout analysis.[1,10,11] In addition, since stress may vary with respect to space (e.g. burial depth) and time, spatial variability and temporal variability also exist.[1] Therefore, the variability of stress is complicated in nature and robust statistical approaches are necessary and prerequisite to fully understand the complexity of stress variability. However, currently, such robust statistical approaches for stress variability characterisation are still lacking.

To alleviate the complexity and make the investigation of stress variability more realistic, assumptions have to be and have already been made,[1] and based on which, many examples of direct statistical processing of stress data can be found in the rock mechanics literature.[6,12–25] For example, it is common to assume that the analysed stresses were obtained within a space and time span that are sufficiently short such that their spatial and temporal variability can be ignored, and the measured stress data are deemed to be practically accurate. Based on these assumptions, several statistical approaches for stress data processing, such as mean stress calculation, statistical distribution model and confidence interval characterisation, have been developed.[12–19] However, in rock mechanics, assessment of stress variability is customarily undertaken by processing principal stress magnitude and orientation separately using scalar- or vector-related statistics (e.g. Fig. 2). This processing effectively decomposes the second order stress tensor into scalar (principal stress magnitudes) and vector (principal

* Corresponding author.
E-mail address: k.gao@mail.utoronto.ca (K. Gao).

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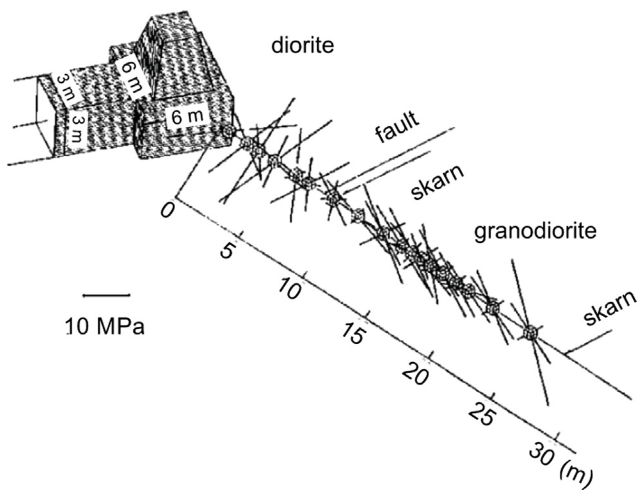
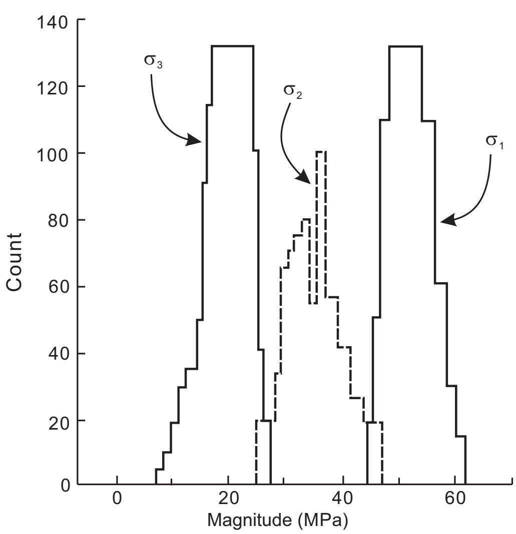
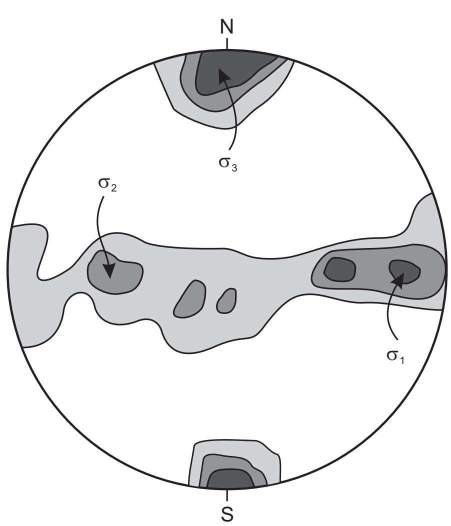


Fig. 1. Dramatic stress change in terms of both principal stress magnitude and orientation observed near a fault (from Obara & Sugawara[7]).



(a) Distribution of principal stress magnitudes



(b) Contouring of principal stress orientations

Fig. 2. Customary analyses of stress examine principal stress magnitude and orientation separately using classical statistics and directional statistics, respectively (after Brady & Brown[34]).

stress orientations) components, to which classical statistics[20] and directional statistics,[21] respectively, are applied. Examples of this approach are widespread in the literature.[6,22–34] All these customary methods not only violate the tensorial nature of stress, but also yield unreasonable results.[16,35–37]

Among many statistics, dispersion (also called scatter, denoting how spread out is a data group) is an effective tool to quantify variability, and it is commonly measured by standard deviation.[20(p.54)] However, standard deviation is only defined for scalar and vector data, and a robust approach to calculating the analogue of standard deviation for stress data is still not clear. This is mainly because of the tensorial nature of stress, which renders classical statistics inapplicable.[35,38] Particularly, for customary applications, when it comes to stress dispersion, one may intuitively calculate the dispersion of principal stress magnitude and orientation separately and hence obtain six dispersions. However, neither the six dispersions individually nor any combination of them gives a sense of the overall stress dispersion. A particular effect of this is that it is currently difficult to quantitatively evaluate overall stress variability, and impossible to quantitatively compare the variability of stress at different engineering sites. To overcome this shortfall and improve the existing related working in rock mechanics,[16,17,19,39,40] based on the above-mentioned assumptions, here we present and examine several dispersion measure approaches, and hence propose a scalar-valued stress dispersion measure for stress variability characterisation.

Rather than customary approaches that analyse principal stress magnitude and orientation separately, in order to remain faithful to the tensorial nature of stress, stress variability analysis should be conducted on the basis of tensor components obtained in a common Cartesian coordinate system. This has been advocated previously by many others.[16–19,39–43] Several researchers have followed this technique in stress dispersion related calculations.[17,19,39,40,42] For example, as dispersion is generally determined relative to the mean, it is necessary to first calculate the mean stress tensor as the mean of the stress tensors referred to a common frame. This approach first takes a group of n stress measurements in a global x-y-z Cartesian coordinate system, the ith stress tensor $\mathbf{S}_i$ of which is given by

$$\mathbf{S}_i = \begin{bmatrix} \sigma_{x_i} & \tau_{xy_i} & \tau_{xz_i} \\ & \sigma_{y_i} & \tau_{yz_i} \\ \text{symmetric} & & \sigma_{z_i} \end{bmatrix}, \tag{1}$$

where σ and τ are the normal and shear tensor components, respectively. The mean stress tensor is then[42]

$$\bar{\mathbf{S}}_{\mathrm{E}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{S}_i = \begin{bmatrix} \bar{\sigma}_x & \bar{\tau}_{xy} & \bar{\tau}_{xz} \\ & \bar{\sigma}_y & \bar{\tau}_{yx} \\ \text{symmetric} & & \bar{\sigma}_z \end{bmatrix} = \frac{1}{n}\begin{bmatrix} \sum_{i=1}^{n}\sigma_{x_i} & \sum_{i=1}^{n}\tau_{xy_i} & \sum_{i=1}^{n}\tau_{xz_i} \\ & \sum_{i=1}^{n}\sigma_{y_i} & \sum_{i=1}^{n}\tau_{yz_i} \\ \text{symmetric} & & \sum_{i=1}^{n}\sigma_{z_i} \end{bmatrix}, \tag{2}$$

where $\bar{\mathbf{S}}_{\mathrm{E}}$ denotes the Euclidean mean stress tensor,[44] and $\bar{\sigma}$ and $\bar{\tau}$ denote the corresponding mean tensor components. A number of reports exist in the literature in which this Euclidean mean has been used as a mean stress tensor.[16,17,19,39–41]

Based on Eq. (2), a so-called stress variance tensor may be calculated.[17,19,39,40] After obtaining the mean stress tensor, a new coordinate system (say, X-Y-Z) is established that coincides with the principal directions of the mean tensor $\bar{\mathbf{S}}_{\mathrm{E}}$, and all the original stress tensors transformed into this new coordinate system. Using the variance function, $\mathrm{var}(x) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, and recognising that $\bar{\tau}_{XY} = \bar{\tau}_{YZ} = \bar{\tau}_{ZX} = 0$, the variance tensor is then calculated as