International Journal of Rock Mechanics & Mining Sciences 79 (2015) 183–192

Contents lists available at ScienceDirect

International Journal of Rock Mechanics & Mining Sciences

journal homepage: www.elsevier.com/locate/ijrmms

# Analytical solutions for the stress of a lined non-circular tunnel under full-slip contact conditions

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ARTICLE INFO

*Article history:*
Received 24 November 2014
Received in revised form
7 May 2015
Accepted 10 August 2015

*Keywords:*
Lined non-circular tunnel
Full-slip contact
Conformal transformation
Power series method
Analytical solutions of stress

ABSTRACT

Stress fields of a lined non-circular tunnel subjected to in situ stress are derived based on the complex variable method and on the assumption that the interface between the liner and surrounding rock is full-slip. The basic equations for solving the stress solutions are obtained according to the stress boundary condition along the inner boundary of the lining and the stress and normal displacement continuity conditions along the rock-lining interface. In the solving process, the support delay is also considered. The basic equations can be solved by the power series method, and the stresses in the surrounding rock mass and lining can be calculated. The distributions of the tangential stresses (also known as the circumferential stresses) along the excavation boundary and the inner boundary of the lining and the contact stresses along the rock-lining interface are analysed. An example demonstrates that the results are significantly affected by the number of terms in the power series. When the number of terms is greater than 100, the boundary conditions can be well satisfied, and the results of the stresses and displacements are highly accurate. The tangential stress results along the inner boundary of the lining for the full-slip condition are compared with those for the perfect bond condition, and the analysis indicates that the maximum value of the tangential stress for the full-slip condition is smaller than that for the perfect bond condition, which gives that the full-slip condition is superior to the perfect bond condition. Thus, the carrying capacity of the lining can be increased if sliding materials are installed between the lining and the surrounding rock mass. The analytic solutions are verified using computer simulation software.



## 1. Introduction

Underground tunnels are widely used in hydropower, traffic, mining and military engineering. To ensure the safety of the tunnels, a concrete lining is applied in these tunnels. The complex variable method developed by Muskhelishvili can be used to calculate the stresses and displacements in the lining and in the surrounding rock mass.[1] This technique is a highly accurate analytical method and is particularly suitable for solving underground tunnel problems.[2] Analytical method has been of high interest for determining stress distribution within lining and surrounding rock mass with high level of accuracy. Recent studies have focused on problems with deep unlined tunnels.[3–6] However, the interaction between the support and the rock mass has been rarely considered in most literatures.

Problems involving lined tunnels cannot be solved easily using the complex variable method. Prior to 2014, only the plane strain problem associated with a single circular tunnel with a ring lining in an infinite domain had been studied in detail.[7–9] Two different regions must be considered when the lining is included, which increases the complexity of the problem.[10,11] Until 2014, the stress and displacement solutions for a closed lined non-circular tunnel were obtained using the Cauchy integration method.[11,12]

In the studies described above, the lining and surrounding rock mass are assumed to have perfect bond contact, i.e., along the rock-lining interface, the stresses and displacements in the normal direction and the shear stresses and displacements in the tangential direction are continuous. However, the contact between the lining and rock cannot be expected to completely satisfy perfect bonding conditions because slippage occurs between the lining and rock. Additionally, the tangential displacements in the lining and rock are also discontinuous because of the slippage.

In this paper, we utilise full-slip contact, i.e., along the interface, the normal stresses and normal displacements are continuous, and the shear stresses are continuous and equal to zero. However, the tangential displacements can be discontinuous. The stress vectors

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http://dx.doi.org/10.1016/j.ijrmms.2015.08.008
1365-1609/

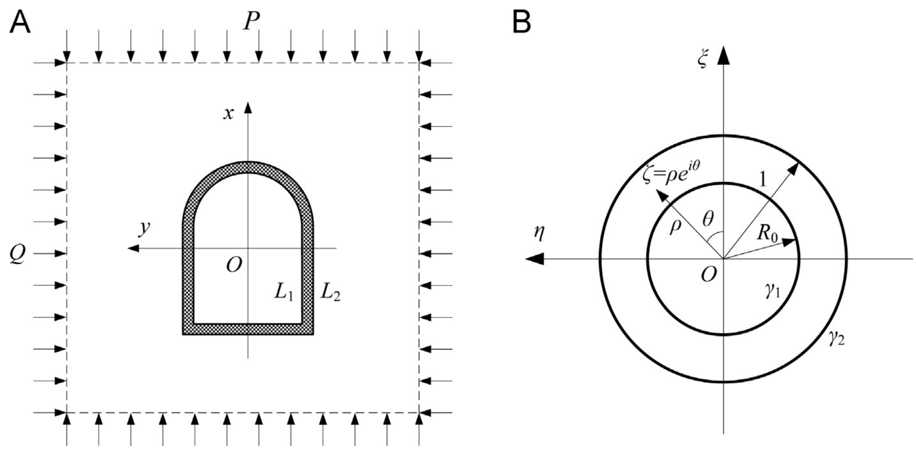


**Fig. 1.** (A) Lined non-circular tunnel under an initial stress field. (B) Ring-shaped region in the ζ plane.

and normal displacements along the interface satisfy the continuity conditions, but slippage can occur between the lining and rock. The full-slip contact condition has been analysed in in-plane and anti-plane problems involving a circular hole with two linings composed of different materials.[13]

Some deformation occurs in the surrounding rock mass before the lining is applied. We assume that this portion of the displacement is given.[12] The stress–displacement solutions for a non-circular tunnel with closed support under full-slip boundary conditions can be obtained by the power series method.

Similar to the studies mentioned above, the following assumptions are made: (1) the surrounding rock mass and lining and the interaction between them behave in a linear elastic manner under the in situ stresses; and (2) the tunnel is deep enough that the case can be simplified as a plane strain problem in an infinite domain (Fig. 1A).

## 2. Fundamental theories and equations

### 2.1. Analytical expressions of displacements in the surrounding rock mass and lining

Because the non-circular tunnel is supported closely and the surrounding rock mass and lining are perfect bond, the interaction can be treated as a typically elastic contact problem. Assuming that the tunnel is sufficiently deep, the problem of a monolithic lining can be treated as a ring-shaped elastomer that is embedded within an infinite domain. The stresses can be analysed using the conformal transformation method of complex functions. The transformation $z = \omega(\zeta)$ is introduced here to transform the complicated support across the section in the $z$ plane (Fig. 1A) into a simpler ring-shaped region in the $\zeta$ plane (Fig. 1B).[2,14] For simplicity, we set the inner and outer radii to $R_0$ (to be determined) and 1, respectively. Let $\gamma_1$ and $\gamma_2$ denote the inner and outer circles, respectively.

If the tunnel is unlined, the maximum displacements in the surrounding rock mass, $u_1^R$ and $v_1^R$, can be written as [2,12]

$$2G_1(u_1^R + iv_1^R) = \kappa_1\varphi_1(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'_1(\zeta)} - \overline{\psi_1(\zeta)} \tag{1}$$

where $\kappa_1 = 3 - 4\mu_1$, $G_1 = E_1/[2(1+\mu_1)]$; $E_1$ and $\mu_1$ are the Young's modulus and Poisson's ratio of the surrounding rock mass, respectively; and $u_1^R$ and $V_1^R$ are the displacement components in the $x$ and $y$ directions, respectively. Let $u_{\rho 1}^R$ and $u_{\theta 1}^R$ denote the normal and tangential displacement components, respectively, in orthogonal curvilinear coordinates in the $z$ plane. From Eq. (1), we obtain [15,16]

$$2G_1(u_{\rho 1}^R + iu_{\theta 1}^R) = \frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}\left[\kappa_1\varphi_1(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'_1(\zeta)} - \overline{\psi_1(\zeta)}\right] \tag{2}$$

We apply the displacement components in orthogonal curvilinear coordinates rather than in rectangular coordinates considered in the literature.[11,12] In this way, the curvilinear coordinate line overlaps with the contact boundary and the equations for full-slip case can be found.

The conformal mapping function $\omega(\zeta)$ and analytic functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ in Eqs. (1) and (2) are given by the following equations [2,12]:

$$z = \omega(\zeta) = R\left(\zeta + \sum_{k=0}^{n} c_k\zeta^{-k}\right) \tag{3}$$

$$\varphi_1(\zeta) = \sum_{k=1}^{n} a_k\zeta^{-k} \tag{4}$$

$$\psi_1(\zeta) = -\frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\varphi'_1(\zeta) + \sum_{k=1}^{n-2} S_k\zeta^k + S'_0 - \frac{pR}{2}(1+\lambda)\zeta^{-1} + \frac{pR}{2}(1-\lambda)\sum_{k=1}^{n} c_k\zeta^{-k} \tag{5}$$

where $\lambda$ is the lateral pressure coefficient and $\lambda = Q/P$. The coefficients $c_k$, $a_k$, $S_k$ and $S'_0$ in Eqs. (3)–(5) are real numbers if the tunnel is symmetric about the $x$-axis.

If the support is installed when the displacement is $\eta$ ($0 \le \eta \le 1$) times the total displacement $u_{\rho 1}^R + iu_{\theta 1}^R$, the displacement that occurs before the support installation is $\eta(u_{\rho 1}^R + iu_{\theta 1}^R)$. The surrounding rock mass and lining interact with each other after the lining is installed. The interaction will restrict the displacement in the surrounding rock mass. This displacement reduced due to the interaction $u_{\rho 2}^R + iu_{\theta 2}^R$ can be derived using the following equation:

$$2G_1(u_{\rho 2}^R + iu_{\theta 2}^R) = \frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}\left[\kappa_1\varphi_2(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'_2(\zeta)} - \overline{\psi_2(\zeta)}\right] \tag{6}$$

where $\varphi_2(\zeta)$ and $\psi_2(\zeta)$ are the analytic functions in the surrounding rock mass caused by the lining support. The power series of these functions can be expressed as

$$\varphi_2(\zeta) = b_0 + \sum_{k=1}^{\infty} b_k\zeta^{-k} \tag{7}$$