International Journal of Rock Mechanics & Mining Sciences 71 (2014) 272–292

Contents lists available at ScienceDirect

International Journal of Rock Mechanics & Mining Sciences

journal homepage: www.elsevier.com/locate/ijrmms

Stresses and displacements in a circular ring under parabolic diametral compression

S.K. Kourkoulis*, Ch.F. Markides

National Technical University of Athens, School of Applied Mathematical and Physical Sciences, Department of Mechanics, 5 Heroes of Polytechnion Avenue, Theocaris Building, Zografou Campus, 157 73 Athens, Greece

ARTICLE INFO

Article history:
Received 4 April 2013
Received in revised form
28 February 2014
Accepted 23 July 2014

Keywords:
Circular ring
Brazilian-disc test
Stress field
Displacement field
Indirect tensile strength
Reflected Caustics

ABSTRACT

The response of a linear elastic circular ring to a parabolic distribution of radial pressure, imposed along two symmetric arcs of its outer perimeter, is explored analytically. Muskhelishvili's complex potentials technique is employed and both the stress- and displacement-fields are obtained in series form. For small holes the ring's loaded arc is assumed identical with that provided by the contact problem of a solid disc squeezed under the same load between the jaws of the ISRM's suggested device for the implementation of the Brazilian-disc test. This assumption is experimentally assessed by properly adapting the Reflected Caustics technique. It is concluded that for inner ring's radius smaller than about thirty per cent of the outer radius, the agreement for the length of the loaded arc between the ring and the solid disc is quite satisfactory. For bigger holes the formulae obtained are still valid assuming however that the boundary conditions are properly modified considering a predefined loaded arc. The variation of stresses and displacements along critical loci of the ring is investigated and is compared with pre-existing solutions for rings under point-load or uniform pressure over arbitrarily chosen loaded arcs. It is concluded that for small holes the type of pressure imposed is not crucial as long as attention is focused at the critical points of the ring where the maximum tensile stress is developed. On the contrary, near the loaded arc the stress field strongly depends on the kind of the externally imposed load which influences also the results at the ring's critical points in case the inner radius increases approaching the outer one. The ratio between the tensile stress at the centre of a solid disc (Brazilian-disc test) and the analogous one at the critical points of a ring loaded under identical conditions is also investigated thoroughly.

1. Introduction

The configuration of a circular ring subjected to either equal compressive line-forces along diametrically opposed generatrices of its lateral surface or to uniform radial pressure along two finite arcs of its periphery (symmetric with respect to the ring's centre), is widely used in various fields of both theoretical and experimental mechanics. Although the respective stress field was studied already since 1910 by Timoshenko [1] and a few years later by Filon [2], various aspects of the problem remain still open. For example, the analytic solution for the displacement field in case a thin ring is subjected to uniform pressure was presented in a convenient easy-to-use form only a few years ago by Tokovyy et al. [3].

The interest of the engineering community on the subject is continuous and undiminished. It has been used, for example, for the analysis of finite strain [4], assessing hot workability [5], determining the collapse load of mild steel rings [6], the stress concentration index of sea-ice [7] etc. Moreover, it was recently proposed as a convenient mean for the determination of fracture toughness without using fatigue pre-cracked specimens [8].

Among the most interesting applications of the ring test is the indirect determination of the tensile strength of brittle materials, especially concrete and various rock-like geomaterials. In fact, the use of ring-shaped specimens instead of solid discs was proposed immediately after the Brazilian-disc test had been introduced by Carneiro [9] and Akazawa [10]. This was done in an attempt to cure some drawbacks of the Brazilian-disc test related to the inevitable stress concentration along the disc–jaw interface which can cause premature local fractures deteriorating the validity of the results obtained by Hondros' classic analysis [11].

The origin of this technique is found in a paper by Ripperger and Davis [12] dated back to 1946. Bortz and Lund [13] and Addinall and Hackett [14] studied the role of the ratio $\rho=R_1/R_2$ (i.e. the ratio of the inner over the outer radii of the ring). Around the same period it was Hobbs [15] who clearly stated that the results of the Brazilian-disc test "*…are suspect because wedge*

* Corresponding author. Tel.: +30 210 7721263; fax: +30 210 7721302.
E-mail address: stakkour@central.ntua.gr (S.K. Kourkoulis).

http://dx.doi.org/10.1016/j.ijrmms.2014.07.009
1365-1609/

Nomenclature

A	the initial curve
A_k	functions of angle ϑ determined from the boundary conditions
a, b	the end points of the caustic curve
b_j, b'_j	the coefficients of the infinite series (they are given in Appendix II)
C	constant depending on the mechanical and optical constants of the material and the caustics experimental set-up
C^*	A global constant equal to $C^*=C/(12\|\lambda_m\|KR_2)$
c_f	the stress optic coefficient of the material. It holds that $c_{f=}\nu/E$
E	Young's modulus of the ring's material
E_f	a fictitious value of Young's modulus
$F(r)$	a function relating the ring's and Brazilian disc's critical transverse stress components along the vertical loading axis
$F(\rho)$	The value of $F(r)$, for $n=1$, at the ring's critical points where the tensile stress is maximized. $F(\rho)=F(r=R_1, n=1)$. It is given in Appendix IV
F_o	the value of $F(\rho)$ for $\rho=0$
$\mathfrak{I}$	the imaginary part of a complex number or function
K	constant equal to $K=(\kappa_1+1)/4\mu_1+(\kappa_2+1)/4\mu_2$
K_H	Hobbs' correction factor $K_H=6+38\rho^2$
ℓ	the semi-contact length in the contact problem
ℓ'	the approximate value of ℓ on the ring, $\ell\approx\ell'=R_2 sin\omega_o$
L_1, L_2	the ring's inner and outer boundaries, respectively
L_3	the cylindrical jaw's boundary (contact problem)
ν	the Poisson's ratio of the ring's material
ν_f	a fictitious value of Poisson's ratio
$\overrightarrow{n_p}$	the unit normal vector at a point P on the deformed ring's front face
P	the arbitrary point on the front face of the ring in the caustics experimental set-up
P'	the projection of P on the reference screen
P_p	the new position of P after the out-of-plane deformation
P_{frame}	the magnitude of the force applied by the loading device
$P(\tau)$	the distribution of the externally applied radial pressure along the contact arc in the contact problem
$P(\vartheta)$	the distribution of the externally applied radial pressure along the loaded arcs of the outer boundary L_2 of the ring
P_c	the maximum value of $P(\vartheta)$
Q	the arbitrary point on the caustic curve on the reference screen
(r,ϑ)	the modulus and argument of the complex variable z
(r_1,ϑ_1) and (r_2,ϑ_2)	the moduli and arguments of the auxiliary complex variables z_1 and z_2, respectively, as they are defined in Fig. 10
r_o	the modulus of the arbitrary point on the caustic's initial curve
$\mathfrak{R}$	the real part of a complex number or function
R_1, R_2	the ring's inner and outer radii, respectively
R_3	the cylindrical jaw's radius, $R_3=1.5R_2$ (contact problem)
t	the ring's (or disc's) thickness
Δt	the thickness change
t_j	the ends of the loaded arcs of the ring's outer boundary L_2
u, v	the Cartesian (horizontal, vertical) displacement components in the ring
u_r, u_ϑ	the polar (radial, transverse) displacement components on the ring
$\vec{w}$	the vector on the reference screen defining the deviation of light
$\vec{W}$	the vector on the reference screen defining the caustic curve
$W_{x'}$, $W_{y'}$	the components of the vector $\vec{W}$ in the $\{O'; x'y',z'\}$ Cartesian reference on the screen
x'_a, y'_a, x'_b, y'_b	The coordinates of the end points a, b, of the caustic curve in the $\{O'; x',y',z'\}$ Cartesian reference on the screen
$\vec{\nabla}$	the gradient operator $\vec{\nabla}=(\partial/\partial x)\vec{i}+(\partial/\partial y)\vec{j}$
z	the complex variable, $z=x+iy=re^{i\vartheta}$
Z_o	the distance between the ring and the reference screen in the Caustics experimental set-up
Z_i	the distance between the focus of the incident light bundle and the ring's front face
α, β	the end points of the initial curve
α_k, α'_k	Complex constants in the Laurent series representation of $\Phi(z)$, $\Psi(z)$
ε	the semi-distance between the end points of the caustic curve
η	the elevation corresponding to the end points α and β of the initial curve
ϑ_{cr}	the critical angle at which the transverse stress component of the ring changes sign on the outer ring's boundary
κ_1, κ_2	Muskhelishvili's constants for the ring and the jaw, respectively
λ_m	the magnification factor of the caustics experimental set-up
μ_1, μ_2	shear moduli for the ring and jaw, respectively
n	the number of terms in the infinite part of the obtained formulae
ρ	the ratio of ring's radii, $\rho=R_1/R_2$
σ_{rr}, $\sigma_{\vartheta\vartheta}$, $\sigma_{r\vartheta}$	the radial, transverse and shear stress components in the ring
$\sigma_{\vartheta\vartheta,A}$, $\sigma_{\vartheta\vartheta,B}$, $\sigma_{\vartheta\vartheta,C}$ and $\sigma_{\vartheta\vartheta,D}$	the transverse stress components at the points A, B, C and D of the ring
$\sigma_{\vartheta\vartheta}^{Ring}$	the transverse stress component of the ring along the loading axis, $\sigma_{\vartheta\vartheta}^{Ring}=\sigma_{\vartheta\vartheta}(r,\vartheta=90^\circ)$
σ_{rr}^{Br}, $\sigma_{\vartheta\vartheta}^{Br}$, $\sigma_{r\vartheta}^{Br}$	the stress components in the Brazilian disc (given in Appendix III)
σ_t	the tensile strength of the material determined by direct tension
σ_t^{Br}	the maximum tensile stress in the Brazilian disc, $\sigma_t^{Br}=\sigma_{\vartheta\vartheta}^{Br}(r=0,\vartheta=90^\circ)$
σ_t^{Ring}	the maximum tensile stress in the ring, $\sigma_t^{Ring}\equiv\sigma_{\vartheta\vartheta}^{Ring}(r=R_1,\ \vartheta=90^\circ)$
σ_y	the yield stress
σ_1, σ_2	the principal stresses
τ	the arc length within $[-\ell,+\ell]$ (contact problem)
τ'	the approximate value of τ on the ring, $\tau\approx\tau'=R_2 cos\vartheta$
$\Phi(z)$, $\Psi(z)$	the complex potentials for the ring
ω_o	the semi-contact angle (semi-loaded arc)
ω	the deviation angle of the reflected light beam in Snell's law