International Journal of Rock Mechanics & Mining Sciences 71 (2014) 381–394

Contents lists available at ScienceDirect

International Journal of Rock Mechanics & Mining Sciences

journal homepage: www.elsevier.com/locate/ijrmms

# Size distribution functions for rock fragments

José A. Sanchidrián [a,*], Finn Ouchterlony [b], Pablo Segarra [a], Peter Moser [b]

[a] Universidad Politécnica de Madrid – ETSI Minas, Madrid, Spain
[b] Montanuniversität Leoben, Leoben, Austria

ARTICLE INFO

*Article history:*
Received 19 October 2013
Received in revised form
7 June 2014
Accepted 6 August 2014
Available online 15 September 2014

*Keywords:*
Rock fragmentation
Weibull
Swebrec
Grady
Re-scaled
Bimodal

ABSTRACT

The capacity of 17 functions to represent the size distribution of fragmented rock is assessed on 1234 data sets of screened fragments from blasted and crushed rock of different origins, of sizes ranging from 0.002 to 2000 mm. The functions evaluated are Weibull, Grady, log-normal, log-logistic and Gilvarry, in their plain, re-scaled and bi-component forms, and also the Swebrec distribution and its bi-component extension. In terms of determination coefficient, the Weibull is the best two-parameter function for describing rock fragments, with a median $R^2$ of 0.9886. Among re-scaled, three-parameter distributions, Swebrec and Weibull lead with median $R^2$ values of 0.9976 and 0.9975, respectively. Weibull and Swebrec distributions tie again as best bi-component, with median $R^2$ of 0.9993. Re-scaling generally reduces the unexplained variance by a factor of about four with respect to the plain function; bi-components further reduce this unexplained variance by a factor of about two to three. Size-prediction errors are calculated in four zones: coarse, central, fines and very fines. Expected and maximum errors in the different ranges are discussed. The extended Swebrec is the best fitting function across the whole passing range for most types of data. Bimodal Weibull and Grady distributions follow, except for the coarse range, where re-scaled forms are preferable. Considering the extra difficulty in fitting a five-parameter function with respect to a three-parameter one, re-scaled functions are the best choice if data do not extend far below 20% passing. If the focus is on the fine range, some re-scaled distributions may still do (Weibull, Swebrec and Grady, with maximum errors of 15–20% at 8% passing), but serious consideration should be given to bi-component distributions, especially extended Swebrec, bimodal Weibull and bimodal Grady.



## 1. Introduction

The assessment of fragmentation by blasting and the subsequent crushing and grinding is an important issue in mining. Operation control and optimization require the description of size distributions in several stages, as fragmentation characteristics influence the mucking productivity, the crusher throughput and energy consumption, the plant efficiency, yield and recovery, and the price itself of the end product in the case of industrial minerals and aggregates. Models, specifications and process data of fragmentation by blasting [1–9], crushers, mills, screens, belts, mineral ducts, bins, cyclones, and plant operation in general [10–20] use distribution functions. However, the domain of sizes or fractions passing in which a given distribution function does really represent the rock fragment size is often neglected. This work tackles this point by examining the errors that can be expected across a broad percent passing range when rock fragmentation is described by different distribution functions.

Rock fragment sizes have been represented for many years, almost exclusively, by means of the Rosin–Rammler (or Weibull) distribution [21–23]; in the last several years, the recently developed Swebrec function [24,25] has gained a relatively high profile as it has been shown to represent the fragmented rock sizes with advantage over the Weibull both in the fines and in the coarse ends. Djordjevic [6] used bimodal Weibull for describing rock fragmented by blasting and Blair [26] studied the behavior of the lognormal and log-logistic bimodals for the same purpose. A wider comparison of functions, namely the Weibull, Swebrec, Grady [27,28] and Gilvarry [29–32], and their bi-component varieties was presented in 2009 [33], with a limited number (28) of data sets. That methodology was applied [34] to a much larger data base, made up of 448 sets of screened fragment size data of rock of a variety of origins and fragmentation processes; the lognormal and its bi-component were also included at that point among the distributions analyzed.

The errors in predicting sizes were determined for each of the distribution functions across the range of data, dividing the passing range in four zones: coarse (>80%), central (80–20%), fine (20–2%) and very fine (<2%). Swebrec was by far the best single component function in all zones, with errors comparable to

* Corresponding author. Tel.: +34 913367060.
*E-mail address:* ja.sanchidrian@upm.es (J.A. Sanchidrián).

http://dx.doi.org/10.1016/j.ijrmms.2014.08.007
1365-1609/

the best bi-components in the coarse and central. Indeed, it is not surprising that a three-parameter function such as the Swebrec outdoes two-parameter ones like Weibull, Grady and lognormal, and raises the question of how the latter ones (for which the independent variable $x$ covers the semi-infinite interval $0 \le x < \infty$) would behave if a re-scaled, finite-interval-transformed form of them (including as third parameter a maximum size, as Swebrec does) were employed. This matter has also been addressed [35] by comparing single, plain functions with their re-scaled variants, and with the Swebrec function itself. The most relevant outcome of that study was that the re-scaled Weibull matched Swebrec as best distributions fitting the four ranges.

The present work extends the comparison to re-scaled vs. bi-component distributions, in an exercise to assess the benefits of the increased number of parameters of the latter, which complicates the fitting problem even with state-of-the-art routines. As work progressed, the experimental data base nearly tripled from the 448 data sets in [34,35], to 1234 in the present paper, thereby improving the statistical significance and giving the conclusions a greater validity.

## 2. The functions

Table 1 lists the distributions tested; their expressions are given below in their cumulative distribution function (CDF) form, $F$.

### 2.1. Plain, single-component:

Weibull–Rosin–Rammler (abbreviated here WRR [21–23]):

$$F_{WRR} = 1 - \exp[-(x/x_c)^n], \quad 0 \le x < \infty \tag{1}$$

where $x_c$ and $n$ are the scale and shape parameters, respectively.

Grady (abbreviated GRA [27,28]):

$$F_{GRA} = 1 - [1 + (x/x_g)^{\alpha}]\exp[-(x/x_g)^{\alpha}], \quad 0 \le x < \infty \tag{2}$$

where $x_g$ and $\alpha$ are the scale and shape parameters, respectively.

Lognormal (LGN):

$$F_{LGN} = \Phi[(\log x - x_m)/s], \quad 0 \le x < \infty \tag{3}$$

where $\Phi$ is the standard normal cumulative distribution; $x_m$ and $s$ are the location and scale parameters (the mean and the standard deviation of the natural logarithm of $x$).

Log-logistic (LGL):

$$F_{LGL} = \frac{1}{1 + (x/x_{50})^{-\gamma}}, \quad 0 \le x < \infty \tag{4}$$

where $x_{50}$ (the median size) and $\gamma$ are the scale and shape parameters, respectively.

Gilvarry (GIL [29–32]):

$$F_{GIL} = 1 - \exp\{-[(x/x_1) + (x/x_1)^2 + (x/x_1)^3]\}, \quad 0 \le x < \infty \tag{5}$$

where $x_1$, $x_2$ and $x_3$ are first, second and third order scale parameters.

### 2.2. Re-scaled

The distributions in Eqs. (1)–(5), which reach the unit value only at infinite size, can be transformed by scaling the abscissa with a maximum size $x_{max}$ and forcing an infinite value of the variable at that point, thereby bringing the function value to 1; this is accomplished by substituting $\xi$ for $x$:

$$\xi = \frac{(x/x_{max})}{1 - (x/x_{max})} = \frac{x}{x_{max} - x}, \quad 0 \le x \le x_{max} \tag{6}$$

The resulting function $F_T(\xi)$, with a semi-infinite ($0 \le \xi < \infty$) domain, is thus transformed into a finite $x$ domain ($0 \le x \le x_{max}$) when expressed in the form $F_T(x)$. As an example, the re-scaled Weibull distribution is, from Eqs. (1) and (6):

$$F_{TWRR} = 1 - \exp[-(\xi/\xi_c)^n] = 1 - \exp\{-[x(x_{max} - x_c)/x_c(x_{max} - x)]^n\}, \quad 0 \le x \le x_{max} \tag{7}$$

Similarly for the other distributions. Re-scaling transformed functions are referred here with the parent function acronym preceded by a T; they have one more parameter ($x_{max}$) than the original function.

One more re-scaled distribution, without a semi-infinite counterpart, is the Swebrec (SWE [24,25]):

$$F_{SWE} = \frac{1}{1 + [\log(x_{max}/x)/\log(x_{max}/x_{50})]^b}, \quad 0 \le x \le x_{max} \tag{8}$$

where $x_{50}$ is the scale parameter (median size) and $b$ a shape parameter.

**Table 1**
Distribution functions.

| Distribution | Domain | No. of parameters | Acronym |
|---|---|---|---|
| Weibull or Rosin–Rammler | Semi-infinite $[0, \infty)$ | 2 | WRR |
| Grady | Semi-infinite $[0, \infty)$ | 2 | GRA |
| Lognormal | Semi-infinite $[0, \infty)$ | 2 | LGN |
| Log-logistic | Semi-infinite $[0, \infty)$ | 2 | LGL |
| Gilvarry | Semi-infinite $[0, \infty)$ | 3 | GIL |
| Re-scaled Weibull | Finite $[0, x_{max}]$ | 3 | TWRR |
| Re-scaled Grady | Finite $[0, x_{max}]$ | 3 | TGRA |
| Re-scaled lognormal | Finite $[0, x_{max}]$ | 3 | TLGN |
| Re-scaled log-logistic | Finite $[0, x_{max}]$ | 3 | TLGL |
| Swebrec | Finite $[0, x_{max}]$ | 3 | SWE |
| Re-scaled Gilvarry | Finite $[0, x_{max}]$ | 4 | TGIL |
| Bi-component Weibull | Semi-infinite $[0, \infty)$ | 5 | BiWRR |
| Bi-component Grady | Semi-infinite $[0, \infty)$ | 5 | BiGRA |
| Bi-component lognormal | Semi-infinite $[0, \infty)$ | 5 | BiLGN |
| Bi-component log-logistic | Semi-infinite $[0, \infty)$ | 5 | BiLGL |
| Extended Swebrec | Finite $[0, x_{max}]$ | 5 | ExSWE |
| Bi-component Gilvarry | Semi-infinite $[0, \infty)$ | 7 | BiGIL |

### 2.3. Bi-component

Bi-component functions, usually representing bi-modal distributions, can be formed as a linear combination of two single-component ones as follows:

$$F_{Bi} = (1-f)F(x, \overline{\pi}_1) + fF(x, \overline{\pi}_2), \quad 0 \le x < \infty \tag{9}$$

where $F$ is any of the functions in Eqs. (1)–(5); $f (0 \le f \le 1)$ is the fraction of the modality with a parameter set $\overline{\pi}_2$, so that $1-f$ is the fraction of the modality $\overline{\pi}_1$. For example, the bi-component log-logistic is

$$F_{BiLGL} = (1-f)F_{LGL} + fF_{LGL} = (1-f)\frac{1}{1 + (x/x_{50,1})^{-\gamma_1}} + f\frac{1}{1 + (x/x_{50,2})^{-\gamma_2}}, \quad 0 \le x < \infty \tag{10}$$

Bi-component functions are referred here with the parent single component function acronym preceded by Bi. The BiWRR was used by Djordjevic [6] in his two-component model of blast fragmentation; the BiLGN and BiLGL have been used by Blair [26].

For the Swebrec case, a so-called extended version (ExSWE) exists [24,25] similar to the bi-components but, like its single component sister, of re-scaled character:

$$F_{ExSWE} = \frac{1}{1 + a[\log(x_{max}/x)/\log(x_{max}/x_{50})]^b + (1-a)[((x_{max}/x) - 1)/((x_{max}/x_{50}) - 1)]^c}, \quad 0 < x \le x_{max} \tag{11}$$

whose behavior at large $x$ is that of Eq. (8) and at small $x$ is that of a power distribution with exponent $c$; $a$ is a partition coefficient similar to $f$ in Eq. (9).