SPEED-GRADIENT ADAPTIVE HIGH-GAIN OBSERVERS FOR SYNCHRONIZATION OF CHAOTIC SYSTEMS

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Abstract: We address the problem of output feedback synchronization of certain chaotic systems, under parameter uncertainty. That is, given a *master* system, the objective is to design a *slave* system that copies the dynamics of the master and reconstructs both the state and the values of the constant parameters of the master system. Hence, the synchronization problem that we address enters in the framework of Pecora and Carroll and relies on adaptive observer theory. In particular, the conditions that we impose take the form of *persistency of excitation*.

Keywords: Synchronization, adaptive observers, chaotic systems

1 Introduction

Since the celebrated paper (Pecora and Carroll 1991) master-slave synchronization of chaotic systems has gained an increasing interest, specifically but not only, due to the applications of this problem into secured communication; see for instance (Shi *et al.* 2004, Amirazodi *et al.* 2002, Kolumban *et al.* 1998) to cite a few. Using chaotic systems to transmit and receive information has several advantages as opposed to more conventional methods relying on periodic carrier signals: 1) chaotic modulation offers a better performance since the correlation of waves is lower than in the case of conventional periodic carriers; 2) it may out-perform conventional methods in the case of noisy channels; 3) chaotic modulation presents robust wideband communications; etc.

In the classic master-slave or, transmitter-receiver scheme, a master circuit is tunned to transmit information using a chaotic carrier signal. The signal is received by a "slave" circuit which, if it can be constructed identically to the master, the information may be decoded out of the chaotic carrier. In practice, it is impossible to repeat the master circuit with the exact values of its components even when these values are known. To this, we add the fact that the information is transmitted through a non-ideal channel. All this uncertainty stymies considerably the faculty of reconstructing the useful information.

In this paper, we present an adaptive approach to synchronization which relies on adaptive observer design. As it has been shown in the important paper (Huijberts and Nijmeijer 1997) the synchronization problem may be recasted in a problem of observer-design, well known in the literature of control systems theory. Different observer-based synchronization schemes have been proposed in the literature, *e.g.* relying on sliding modes: (Boutat-Baddas *et al.* 2004); high-gain: (Amirazodi *et al.* 2002); Luenberger-based observers: (Boutayeb *et al.* 2002), *etc.* We propose an adaptive observer for a class of systems that covers certain chaotic systems. Then, we give sufficient conditions to achieve master-slave synchronization in the event of parameter uncertainty and assuming that only an output – possibly part of the master's state – is available for measurement.

The rest of the paper is organized as follows. In coming section we introduce some notation and definitions of stability that set the framework for our main results. In Section 3 we present an adaptive observer for a class of detectable systems and give examples of chaotic systems that fit in our framework. In Section 5 we present the proofs of our findings, before concluding with some remarks.

2 Preliminaries

Notation. We say that a function $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathcal{A}$ with $\mathcal{A}$ a closed, not necessarily compact set, satisfies the basic regularity assumption (BRA) if $\phi(t, \cdot)$ is locally Lipschitz uniformly in t and $\phi(\cdot, x)$ is measurable. We denote the usual Euclidean norm of vectors by $|\cdot|$ and use the same symbol for the matrix induced norm. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\alpha \in \mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$, $\beta(s, \cdot)$ is strictly decreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. We denote the solution of a differential equation $\dot{x} = f(t, x)$ starting at $x_\circ$ at time $t_\circ$ by $x(\cdot, t_\circ, x_\circ)$; furthermore, if the latter are defined for all $t \geq t_\circ$ we say that the system is forward complete.

Definition 1 (Uniform global stability) The origin of

$$\dot{x} = f(t, x) \tag{1}$$

where $f(\cdot, \cdot)$ satisfies the BRA, is said to be uniformly globally stable (UGS) if there exists $\kappa \in \mathcal{K}_\infty$ such that, for each $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, each solution $x(\cdot, t_\circ, x_\circ)$ of (1) satisfies

$$|x(t, t_\circ, x_\circ)| \leq \kappa(|x_\circ|) \qquad \forall\, t \geq t_\circ\,. \tag{2}$$

Definition 2 (Uniform global asymptotic stability) The origin of (1) is said to be uniformly globally asymptotically stable (UGAS) if it is UGS and uniformly globally attractive, *i.e.*, for each pair of strictly positive real numbers (r, σ), there exists $T > 0$ such that for each solution

$$|x_\circ| \leq r \implies |x(t, t_\circ, x_\circ)| \leq \sigma \qquad \forall\, t \geq t_\circ + T\,.$$

Definition 3 (UES) The origin of the system $\dot{x} = f(t, x)$ is said to be uniformly exponentially stable on any ball if for any $r > 0$ there exist two constants k and $\gamma > 0$ such that, for all $t \geq t_\circ \geq 0$ and all $x_\circ \in \mathbb{R}^n$ such that $|x_\circ| < r$

$$|x(t, t_\circ, x_\circ)| \leq k\, |x_\circ|\, e^{-\gamma(t-t_\circ)}\,. \tag{3}$$

Definition 4 (USPAS) The origin of (1) is said to be uniformly semiglobally practically asymptotically stable (USPAS) if for each positive real numbers $\Delta > \delta > 0$ and $\sigma > 0$ there exist $T > 0$ and $\kappa \in \mathcal{K}_\infty$ such that $|x(t, t_\circ, x_\circ)| \leq \kappa(|x_\circ|)$ for all $t \geq t_\circ \geq 0$ and

$$|x_\circ| \leq \Delta \implies |x(t, t_\circ, x_\circ)| \leq \sigma + \delta \qquad \forall\, t \geq t_\circ + T\,.$$

3 Adaptive Observers With Persistency of Excitation

Consider a nonlinear system of the form

$$\dot{x} = A(y)x + \Psi(x)\theta + B(t, x) \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector; $\theta \in \Theta$ is a vector of unknown constant parameters and Θ is a compact of $\mathbb{R}^m$; $y = Cx$ is a measurable output; the pair $(A(y(t)), C)$ is detectable, that is $y(t) \equiv 0$ implies that $x(t) \to 0$; the functions Ψ and B are globally Lipschitz, *i.e.* there exist ψ_M and b_M such that, for any vectors $\zeta \in \mathbb{R}^m$, with $|\zeta| = 1$, x_1, $x_2 \in \mathbb{R}^n$ and all $t \geq 0$,

$$|\Psi(x_1)\zeta - \Psi(x_2)\zeta| \leq \psi_M\, |x_1 - x_2| \tag{5a}$$
$$|B(t, x_1) - B(t, x_2)| \leq b_M\, |x_1 - x_2|\,. \tag{5b}$$

Moreover, there exists $\psi_0 \geq 0$ such that

$$\max_{|\zeta|=1} \left|\zeta^\top \Psi(0)\zeta\right| \leq \psi_0\,. \tag{6}$$

Under these conditions we propose for systems of the form (4), the adaptive observer

$$\dot{\hat{x}} = A(y)\hat{x} - L(t, y)C(x - \hat{x}) + B(t, \hat{x}) + \Psi(\hat{x})\hat{\theta} \tag{7}$$

where $L(\cdot, \cdot)$ satisfies the basic regularity assumption. Using (4), and defining $\bar{x} := \hat{x} - x$, $\bar{\theta} := \hat{\theta} - \theta$ the estimation error dynamics is given by

$$\dot{\bar{x}} = [\,A(y) - L(t, y)C\,]\bar{x} + \Psi(\bar{x} + x(t)) \times\bar{\theta} + \Phi(t, \bar{x}, x(t), \theta) \tag{8a}$$
$$\Phi(t, \bar{x}, x(t), \theta) := [\,\Psi(\bar{x} + x(t)) - \Psi(x(t))\,]\theta + B(t, \bar{x} + x(t)) - B(t, x(t))\,. \tag{8b}$$

Conditions (5) and the assumption that $\theta \in \Theta$ where Θ is a compact of appropriate dimension imply that there exists $\theta_M > 0$ such that

$$|\Phi(t, \bar{x}, x(t), \theta)| \leq \psi_M\theta_M\, |\bar{x}| + b_M\, |\bar{x}| =: \phi_M\, |\bar{x}|\,. \tag{9}$$

The following assumption on the observer gain L guarantees that the state estimation errors tend uniformly to zero; roughly the condition is that the gain L, through the measurable output $y(t)$, makes the error dynamics persistently excited.

Assumption 1 Define $y_t := y(t)$ for each t. There exists a globally bounded positive definite matrix function $P(\cdot)$ such that $p_M \geq |P|$ and, defining $\bar{A}(t, y_t) := A(y_t) - L(t, y_t)C$, $-Q(t, y_t) := \bar{A}(y_t)^\top P(t) + P(t)^\top \bar{A}(y_t) + \dot{P}(t)$ we have the following for all $t \geq 0$ and all $y_t \in \mathbb{R}^m$

1. $Q(t, y_t) \geq 0$
2. There exist μ and $T > 0$ such that

$$\int_t^{t+T} Q(\tau, y_\tau)d\tau \;\geq\; \mu I > 0\,, \qquad \forall\, t \geq 0 \tag{10}$$

3. There exists $q_M > 0$ such that $q_M \geq |Q(t, y_t)|$.