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Technical Note

Unrealistic statistics: How average constitutive coefficients can produce non-physical results

*Daniel Robertson**, *Douglas Cook*[1]

Department of Mechanical Engineering, New York University-Abu Dhabi, PO BOX 129188, Abu Dhabi, United Arab Emirates

A R T I C L E I N F O

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A B S T R A C T

The coefficients of constitutive models are frequently averaged in order to concisely summarize the complex, nonlinear, material properties of biomedical materials. However, when dealing with nonlinear systems, average inputs (e.g. average constitutive coefficients) often fail to generate average behavior. This raises an important issue because average nonlinear constitutive coefficients of biomedical materials are commonly reported in the literature. This paper provides examples which demonstrate that average constitutive coefficients applied to nonlinear constitutive laws in the field of biomedical material characterization can fail to produce average stress–strain responses and in some cases produce non-physical responses. Results are presented from a literature survey which indicates that approximately 90% of tissue measurement studies that employ a nonlinear constitutive model report average nonlinear constitutive coefficients. We suggest that reviewers and editors of future measurement studies discourage the reporting of average nonlinear constitutive coefficients. Reporting of individual coefficient sets for each test sample should be considered and discussed as designation for a "best practice" in the field of biomedical material characterization.

1. Introduction

The mechanical response of biological tissues is typically nonlinear (Sacks and Sun, 2003) and highly variable (Cook et al., 2014). Nonlinear constitutive laws (e.g. Fung (1990), Ogden et al., (2004), (Ferruzzi et al., 2011)) are frequently used describe these complex materials. To quantify the degree of variability observed in biomaterials, statistical measures of central tendency and distribution (e.g. mean, median, and standard deviation) of individual nonlinear constitutive coefficients are typically reported. However, studies conducted in other fields have demonstrated that average input coefficients can fail to produce average results or outputs (Golowasch et al., 2002). From a purely mathematical standpoint it is known that average inputs to nonlinear systems do not generally produce average outputs (Westfall and Henning, 2013). Symbolically this can be written as follows:

$$E(f(X,Y)) \neq f(E(X), E(Y)) \quad (1)$$

*Corresponding author. Tel.: +971 26285229.
E-mail addresses: djr7@nyu.edu (D. Robertson), dc125@nyu.edu (D. Cook).
[1]Tel.: +971 26284579.

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where $\mathbf{E}$ represents the expectation (or average), f is a nonlinear function, and X, Y are random variables (e.g. constitutive coefficients). Stated more simply, except under very limited conditions, the average mechanical response of tissue samples in any given study may not equal the response produced when applying average constitutive coefficients to the underlying nonlinear constitutive law.

This raises a very important issue because (as will be shown shortly) average nonlinear constitutive coefficients are routinely reported in the literature. This practice assumes that the average(s) of coefficient sets will produce an average material response. The current manuscript gives several examples which demonstrate that such an assumption is incorrect, and also illustrates the types of errors that are produced when average constitutive coefficients are applied to nonlinear constitutive laws in the field of biological tissue characterization. In addition, a survey of the literature was performed to assess the frequency with which average input coefficients are reported in the field of biological tissue characterization. The overarching goal of the study is to raise awareness of this important issue among the research community and to demonstrate the inability of average coefficients to capture average behavior. Finally, we conclude with recommendations of best practices for reporting the material response of nonlinear biomedical materials.

2. Theory

Errors resulting from the process of averaging input coefficients of nonlinear models ('averaging failure') arise from several sources, which are described in this section. The first source of averaging failure can be demonstrated using probability theory. For linear functions such as $aX+b$, where a and b are constants and X is a random variable the expected (average) value of $aX+b$ is equal to a times the expected value of X plus b: $\mathbf{E}(aX+b)=a\mathbf{E}(X)+b$. This is known as the "homogeneity property". More generally letting $g(X,Y)=aX+bY$ we can write $\mathbf{E}(g(X,Y))=g(\mathbf{E}(X),\mathbf{E}(Y))$. This is commonly known as the "additive property". These two properties confirm that the expected value of a *linear function* is indeed equal to the *linear function* evaluated at the expected or average value of its input arguments. Thus reporting average constitutive parameters for linear models does not present any immediate complications. However, this is only true of linear functions. If, for example, $h(X)=e^X$, then $\mathbf{E}(h(X))\neq h(\mathbf{E}(X))$. In the special case that a nonlinear function is either entirely convex or entirely concave the direction of error (but not the magnitude of error) can be predicted using Jensen's inequality which states:

- If $f(Y)$ is a convex function, then $\mathbf{E}(f(Y))>f(\mathbf{E}(Y))$.
- If $f(Y)$ is a concave function, then $\mathbf{E}(f(Y))<f(\mathbf{E}(Y))$.

Stated more directly, inserting average or expected coefficient values into nonlinear constitutive models does not generally produce an average or expected outcome. The preceding arguments can be verified in standard probability theory textbooks (Billingsley, 1995; Westfall and Henning, 2013).

The second source of averaging failure involves the structure of certain nonlinear models. One approach in modeling biological tissues is to use segmented or piecewise constitutive laws in which two or more curves are connected together to capture observed tissue behavior. Such models require that specific conditions are met at the interface between adjacent regions (typically C^0 and C^1 continuity). Experimental samples that produce data that is appropriate for such models naturally satisfy these continuity requirements, and the resulting coefficients therefore also satisfy these requirements. However, the process of averaging coefficients across samples neglects these important relationships. As such, the average of individual coefficient sets may not satisfy such continuity requirements.

Third, averaging can cause problems when the underlying distribution of each coefficient is not well understood. For example, while the average of a normal distribution is an appropriate measure of central tendency, the average of a lognormal (skewed) distribution is not an appropriate measure of central tendency because samples drawn from the long tail of the distribution will shift the average in the direction of the tail. For distributions that are not normal, the median of the data sample is often a more appropriate measure of central tendency than the average. The choice of which statistic to use when reporting central tendency is often complicated because the distributions of model coefficients are difficult to assess with sample sizes that are typical in our community (~5–20 samples).

3. Methods

A survey of the literature was performed to observe the manner in which nonlinear stress–strain curves of biomedical materials are reported in the literature. Candidate articles were found by searching the Pubmed and Science Direct databases for the following keywords: nonlinear, mechanical properties, biological tissue. Only articles which employed a nonlinear constitutive model (e.g. Fung, Ogden, exponential fiber laws etc.) were included in the current study. Studies which did not give a clear mathematical description or implementation of the constitutive model being used were excluded. A total of 28 articles were investigated in the current study.

Data from the literature survey was used in two ways: the first, to assess reporting trends in our community, and the second, to determine the types of errors that can be produced when employing average nonlinear constitutive coefficients. Analysis of this nature included digitizing stress–strain curves from previous studies, calculating averages in cases when averages were not reported, and creating new curves using random sampling based on reported averages and standard deviations.

4. Results

Average nonlinear constitutive coefficients were reported in 86% of papers included in the literature survey (24 of 28 studies). Approximately one third of the papers (39%) reported only