JOURNAL OF THE MECHANICAL BEHAVIOR OF BIOMEDICAL MATERIALS 29 (2014) 20–32

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Research Paper

A novel strategy to identify the critical conditions for growth-induced instabilities

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ARTICLE INFO

Article history:
Received 11 April 2013
Received in revised form
6 August 2013
Accepted 13 August 2013
Available online 29 August 2013

Keywords:
Growth
Morphogenesis
Residual stress
Prestress
Stability analysis
Instabilities

ABSTRACT

Geometric instabilities in living structures can be critical for healthy biological function, and abnormal buckling, folding, or wrinkling patterns are often important indicators of disease. Mathematical models typically attribute these instabilities to differential growth, and characterize them using the concept of fictitious configurations. This kinematic approach toward growth-induced instabilities is based on the multiplicative decomposition of the total deformation gradient into a reversible elastic part and an irreversible growth part. While this generic concept is generally accepted and well established today, the critical conditions for the formation of growth-induced instabilities remain elusive and poorly understood. Here we propose a novel strategy for the stability analysis of growing structures motivated by the idea of replacing growth by prestress. Conceptually speaking, we kinematically map the stress-free grown configuration onto a prestressed initial configuration. This allows us to adopt a classical infinitesimal stability analysis to identify critical material parameter ranges beyond which growth-induced instabilities may occur. We illustrate the proposed concept by a series of numerical examples using the finite element method. Understanding the critical conditions for growth-induced instabilities may have immediate applications in plastic and reconstructive surgery, asthma, obstructive sleep apnoea, and brain development.

1. Introduction

Structural instabilities in the form of creases, folds, or wrinkles are inherent to living matter. In many living systems, the formation of structural instabilities is critical to biological function, e.g., to increase the surface-to-volume ratio of the system (Wyczalkowski et al., 2012). Typical examples are wrinkling of skin (Buganza Tepole et al., 2011), villi formation in the intestine (Balbi and Ciarletta, 2013), and folding of the developing brain (Xu et al., 2010). In other biological systems, however, the formation of structural instabilities can be a critical hallmark of disease, e.g., when associated with a narrowing lumen. The most prominent example of this latter category is the folding of the mucous membrane in asthmatic airways (Wiggs et al., 1997). It is thus not surprising that the mathematical modeling of folding in tubular organs (Ciarletta and Ben Amar, 2012), in particular the modeling of the folding mucous membrane (Moulton and Goriely, 2011; Li et al., 2011; Xie et al., 2013), has drawn increasing scientific attention within the past decade.

Continuum approaches toward the formation of geometric instabilities in living systems typically adopt the concept of

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1751-6161/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.jmbbm.2013.08.017

finite growth (Rodriguez et al., 1994). This kinematic approach toward growth is based on the introduction of a fictitious growth configuration (Garikipati, 2009) and on the multiplicative decomposition of the deformation gradient into a reversible elastic and an irreversible growth part (Taber, 1995). Mathematically speaking, in this approach, growth is represented through a second order, isotropic (Himpel et al., 2005), transversely isotropic (Zöllner et al., 2012), orthotropic (Göktepe et al., 2010), or generally anisotropic growth tensor. As discussed in a recent review article on growth (Ambrosi et al., 2011), the evolution of this growth tensor is typically either morphogenetically driven (Li et al., 2012) or mechanically driven (Menzel and Kuhl, 2012).

Here we focus on morphogenetically driven growth and on its role in the formation of structural instabilities. The first rigorous mathematical analyses of growth-induced morphogenetic instabilities studied the failure models of a shrinking spherical shell (Goriely and BenAmar, 2005) and of a growing spherical shell under external pressure (Ben Amar and Goriely, 2005). Motivated by the clinical problem of mucosal folding during chronic airway wall remodeling, recent studies explored the buckling of single-layered (Moulton and Goriely, 2011) and double-layered (Cao et al., 2012; Jin et al., 2011) hollow cylindrical tubes. In realistic airway wall geometries, the thickness of the folding inner layer is typically orders of magnitude smaller than the cylindrical airway structure itself. Accordingly, a recent study suggested to model mucosal folding using the concept of surface growth (Papastavrou et al., 2013), an approach for which the mucosal surface itself is equipped with its own potential energy (Steinmann, 2008). A similar approach was recently proposed for the longitudinal growth in double-layered cylinders (Vandiver and Goriely, 2009) to simulate the effects of surface growth in plants (Holland et al., in press).

Inhomogeneous growth induces a state of prestrain or residual stress (Fung, 1991). Residual stress, which must not be confused with prestress in this paper, is the stress in a body in the unloaded configuration (Menzel, 2005; Rausch et al., in press; Rausch and Kuhl, 2013). In the following we distinguish between two geometrically identical reference configurations, one *stress-free* and one *pre-stressed*. We assume that the reverse (elastic) deformation from the stress-free grown configuration to the pre-stressed reference configuration results in pre-stress. The theory of elasticity for a body under initial stress (prestress here) was first established by Biot (1939). More recently, Johnson and Hoger (1993) studied the dependence of the elasticity tensor on residual stress where the residual stress was produced by an elastic deformation (Hoger, 1986; Marlow, 1992; Hoger, 1993). The problem of dead loading in the mathematical theory of linear elasticity with initial stress (Man and Carlson, 1994) bears certain similarities to our analysis here. Ogden (1992) details on stability and uniqueness of solution of incremental boundary value problem. In Ogden (2003), he employed his generic method to soft tissues and tube extension/inflation problems with residual stresses arising from a uniform circumferential stress. For further mathematical details of symmetry, bifurcation, and instabilities in the context of elasticity with a prestressed reference configuration we refer to Bharatha and Levinson (1978), Capriz and Guidugli (1979), and Wan and Marsden (1983).

In the theory of linear elasticity, a condition for the existence of solutions is that incremental deformations require positive energy. This physically motivated condition translates into the mathematically motivated condition of pointwise stability. In the classical theory of elasticity the pointwise stability condition corresponds to the positive definiteness of the constitutive tensor. However, in the context of prestress, we cannot simply adopt this classical pointwise stability condition for two reasons: (i) The condition that the elasticity tensor is positive definite may no longer be feasible; and (ii) The pointwise stability condition does not directly render the positive definiteness of the constitutive tensor, since there are several elasticity tensors, all of which are functions of the prestress that describe prestressed materials (see similar discussions in Hoger, 1995, for the case of residual stress). In this paper, we re-establish the governing equations for prestressed continua and *properly* impose the pointwise stability condition.

The necessary and sufficient conditions for the loss of well-posedness of the boundary value problem for linear elastic, homogeneous continua are the loss of strong ellipticity of the governing equations and the boundary complementing condition (see e.g. Simpson and Spector, 1985; Benallal et al., 1993). A sufficient condition for stability of elastic continua is the pointwise stability criterion (Hill, 1957). A general theory of uniqueness and stability for elasto-plastic solids was given by Hill (1958). The propagation of surface waves in bodies has been investigated in Dowaikh and Ogden (1990). Bifurcation in the form of surface instabilities has been investigated for arbitrary nonlinear elastic materials under conditions of an equibiaxial prestress (Reddy, 1982), and for plane strain (Reddy, 1983). A comprehensive study on uniqueness, loss of ellipticity, and localization for the time-discrete, rate-dependent boundary value problems with softening can be found in Benallal et al. (2010).

In view of these considerations, the goal of this contribution is to explore the critical conditions for growth-induced instabilities in living structures. In Section 2, we illustrate the kinematics of finite growth based on the multiplicative decomposition of the total deformation gradient into an elastic and a growth part. In Section 3, we introduce the key idea of this work, the conceptual replacement of this growth part by prestress. In Section 4, we discuss the condition for strong ellipticity, the condition for pointwise stability, and the boundary complementing condition in the context of prestress. In Section 5, we illustrate these three conditions for a simple homogeneous model problem, and for the inhomogeneous problems of growth and shrinkage of a hollow cylinder and of a solid sphere. We conclude with a critical discussion and an outlook in Section 6.

1.1. *Notation and definitions*

The three-dimensional Euclidean space is denoted $\mathbb{E}^3$. Direct notation is adopted throughout. Occasional use is made of index notation, the summation convention for repeated indices being implied. The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e., the single contraction, is denoted $\boldsymbol{a}\cdot\boldsymbol{b}=[\boldsymbol{a}]_i[\boldsymbol{b}]_i$. The scalar product of two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, i.e., the double contraction, is denoted $\boldsymbol{A}:\boldsymbol{B}=[\boldsymbol{A}]_{ij}[\boldsymbol{B}]_{ij}$. The action of a second-order tensor $\boldsymbol{A}$ on a vector $\boldsymbol{a}$ is understood as $[\boldsymbol{A}\cdot\boldsymbol{a}]_i=[\boldsymbol{A}]_{ij}[\boldsymbol{a}]_j$ and $[\boldsymbol{a}\cdot\boldsymbol{A}]_i=[\boldsymbol{a}]_j[\boldsymbol{A}]_{ji}$. The double contraction of a third-order tensor $\boldsymbol{C}$ and a second-order tensor $\boldsymbol{B}$ renders a vector according to $[\boldsymbol{C}:\boldsymbol{B}]_i=[\boldsymbol{C}]_{ijk}[\boldsymbol{B}]_{jk}$. The action of a third-order tensor $\boldsymbol{C}$ on a vector $\boldsymbol{a}$, denoted $\boldsymbol{C}\cdot\boldsymbol{a}$, is a second-order tensor with components $[\boldsymbol{C}\cdot\boldsymbol{a}]_{ij}=[\boldsymbol{C}]_{ijm}[\boldsymbol{a}]_m$.