ARTICLE IN PRESS

Ain Shams Engineering Journal (2015) **xxx**, xxx–xxx

Ain Shams University

Ain Shams Engineering Journal

www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

An exponentially fitted finite difference scheme for a class of singularly perturbed delay differential equations with large delays

P. Pramod Chakravarthy [a,*], S. Dinesh Kumar [a], R. Nageshwar Rao [b]

[a] *Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India*
[b] *School of Advanced Sciences, VIT University, Vellore, Tamil Nadu 632014, India*

Received 23 February 2015; revised 21 August 2015; accepted 16 September 2015

KEYWORDS

Singular perturbations;
Boundary layer;
Delay differential equation;
Exponentially fitted finite difference method

Abstract This paper deals with singularly perturbed boundary value problem for a linear second order delay differential equation. It is known that the classical numerical methods are not satisfactory when applied to solve singularly perturbed problems in delay differential equations. In this paper we present an exponentially fitted finite difference scheme to overcome the drawbacks of the corresponding classical counter parts. The stability of the scheme is investigated. The proposed scheme is analyzed for convergence. Several linear singularly perturbed delay differential equations have been solved and the numerical results are presented to support the theory.

1. Introduction

Singular perturbation problems arise very frequently in fluid dynamics, elasticity, aerodynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion. An overview of some existence and uniqueness results and applications of singularly perturbed equations may be obtained from [1–4]. Various approaches to the design and analysis of approximate numerical methods for singularly perturbed differential equations can be found in [5–8] and the references cited in them.

Delay differential equations arise widely in various application fields and are also described in technical devices like control circuits. Nowadays the delay differential equations are ubiquitous in various branches of bioscience and control theory: ecology, chemostat systems, epidemiology, immunology, compartmental studies, neutral network and the navigational control of ships and aircraft (with respectively large and short lags) and in more general control problems [9–21]. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then to react to it. Delay differential equations are of the retarded type if the delay argument does not occur in the highest order term. If we restrict this class in which the highest order term is multiplied by a small parameter, then we get singularly perturbed delay differential equations of

* Corresponding author. Tel.: +91 712 2801404.
E-mail addresses: pramodpodila@yahoo.co.in (P. Pramod Chakravarthy), mathdinesh005@gmail.com (S. Dinesh Kumar), nrao_ragi@yahoo.co.in (R. Nageshwar Rao).
Peer review under responsibility of Ain Shams University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.asej.2015.09.004
2090-4479 © 2015 Production and hosting by Elsevier B.V. on behalf of Ain Shams University.

retarded type. The numerical study of second order singularly perturbed differential-difference equation with small shift or delay has been given in [22–38] and references therein.

Amiraliyev and Cimen [39] have given an exponentially fitted difference scheme on a uniform mesh for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term. Subburayan and Ramanujam [40] presented an initial value technique to solve singularly perturbed boundary value problem for the second order ordinary differential equations of convection–diffusion type with a delay. They also developed an asymptotic initial value technique to solve singularly perturbed boundary value problem for the second order ODE with the discontinuous convection–diffusion coefficient term [41].

In this paper we present an exponentially fitted finite difference scheme to solve singularly perturbed delay differential equation of second order with a large delay. For many singular perturbation problems a reduced problem is well defined and known a priori. We used Runge–Kutta method to solve the reduced problem. A condition at $x = 1$ is obtained by approximating the solution at $x = 1$ by the solution of the reduced problem at the same point. A fitting factor is introduced in a finite difference scheme and is obtained from the theory of singular perturbations. The stability of the scheme is investigated. The proposed scheme is analyzed for convergence. Several linear singularly perturbed delay differential equations have been solved and the numerical results are presented to support the theory.

2. Exponentially fitted finite difference method

To describe the method, we consider a singularly perturbed boundary value problem for the delay differential equation of the form:

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x-1) = f(x), x \in [0,2] \tag{1}$$

subject to the interval and boundary conditions,

$$\begin{aligned} &y(x) = \phi(x), x \in [-1, 0]; \\ &y(2) = \beta, \end{aligned} \tag{2}$$

where $0 < \varepsilon << 1$ and $a(x) \geqslant \alpha > 0, \theta \leqslant b(x) < 0, a(x), b(x), f(x)$ are given sufficiently smooth functions on $[0,2]$, $\phi(x)$ is smooth function on $[-1,0]$ and β is a given constant which is independent of ε. For small values of ε, the boundary value problem (1) along with (2) exhibits a strong boundary layer at $x = 2$ (c.f. [12]).

The linear ordinary differential equation (1) cannot, in general, be solved analytically because of the dependence of $a(x), b(x)$ and $f(x)$ on the spatial coordinate x. We divide the interval $[0,2]$ into $2N$ equal parts with constant mesh length h. Let $0 = x_0, x_1, \ldots, x_N = 1, x_{N+1}, x_{N+2}, \ldots, x_{2N} = 2$ be the mesh points. Then we have $x_i = ih; i = 0, 1, 2, \ldots, 2N$. If we consider, the interval $[x_{i-1}, x_{i+1}]$ and the coefficients of Eq. (1) are evaluated at the midpoint of each interval, then we will obtain the differential equation

$$-\varepsilon y''(x) + a_i y'(x) = f_i - b_i y(x_i - 1), \quad x_{i-1} < x < x_{i+1}. \tag{3}$$

The analytical solution of Eq. (3) is of the form

$$y(x) = A_i + B_i e^{\frac{a_i(x-x_i)}{\varepsilon}} + \frac{x - x_i}{a_i}(f_i - b_i y(x_i - 1)), \quad x_{i-1} < x < x_{i+1}, \tag{4}$$

where A_i, B_i are arbitrary constants. We obtain the arbitrary constants A_i and B_i using the conditions $y(x_{i-1}) = y_{i-1}, y(x_i) = y_i, y(x_{i+1}) = y_{i+1}$.

From (4) we have,

$$y_{i-1} = A_i + B_i e^{\frac{-a_i h}{\varepsilon}} - \frac{h}{a_i}[f_i - b_i y(x_i - 1)], \tag{5}$$

$$y_{i+1} = A_i + B_i e^{\frac{a_i h}{\varepsilon}} + \frac{h}{a_i}[f_i - b_i y(x_i - 1)] \text{ and} \tag{6}$$

$$y_i = A_i + B_i. \tag{7}$$

Now, we have

$$y_{i+1} - 2y_i + y_{i-1} = B_i\left(e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}\right).$$

Therefore,

$$B_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}}. \tag{8}$$

Since, $A_i = y_i - B_i$, we have

$$A_i = \frac{\left(e^{\frac{a_i h}{\varepsilon}} + e^{\frac{-a_i h}{\varepsilon}}\right)y_i - (y_{i+1} + y_{i-1})}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}}. \tag{9}$$

Substituting (8) and (9) in (4), we get

$$y(x) = \frac{\left(e^{\frac{a_i h}{\varepsilon}} + e^{\frac{-a_i h}{\varepsilon}} - 2e^{\frac{a_i(x-x_i)}{\varepsilon}}\right)y_i + \left(e^{\frac{a_i(x-x_i)}{\varepsilon}} - 1\right)(y_{i+1} + y_{i-1})}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}} + \frac{(x - x_i)}{a_i}[f_i - b_i y(x_i - 1)]. \tag{10}$$

From (10), we have

$$y(x_{i-1}) = \frac{\left(e^{\frac{a_i h}{\varepsilon}} + e^{\frac{-a_i h}{\varepsilon}} - 2e^{\frac{-a_i h}{\varepsilon}}\right)y_i + \left(e^{\frac{-a_i h}{\varepsilon}} - 1\right)(y_{i+1} + y_{i-1})}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}} - \frac{h}{a_i}[f_i - b_i y(x_i - 1)],$$

$$y(x_{i+1}) = \frac{(e^{\frac{a_i h}{\varepsilon}} + e^{\frac{-a_i h}{\varepsilon}} - 2e^{\frac{a_i h}{\varepsilon}})y_i + [e^{\frac{a_i h}{\varepsilon}} - 1](y_{i+1} + y_{i-1})}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}} + \frac{h}{a_i}[f_i - b_i y(x_i - 1)].$$

Therefore, we have

$$y(x_{i+1}) - y(x_{i-1}) = \frac{y_{i+1} - 2y_i + y_{i-1}}{e^{\frac{a_i h}{\varepsilon}} - 2 + e^{\frac{-a_i h}{\varepsilon}}}\left(e^{\frac{a_i h}{\varepsilon}} - e^{\frac{-a_i h}{\varepsilon}}\right) + \frac{2h}{a_i}[f_i - b_i y(x_i - 1)]. \tag{11}$$

Now, to find the solution in $[0,2]$, we consider the second order finite difference scheme

$$-\varepsilon\sigma(\rho)\left(\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}\right) + a(x_i)\left(\frac{y(x_{i+1}) - y(x_{i-1})}{2h}\right) = f(x_i) - b(x_i)y(x_i - 1) + O(h^2), 1 \leqslant i \leqslant 2N - 1 \tag{12}$$

where $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of (12) converges uniformly to the solution of (1), (2) and $\rho = \frac{h}{\varepsilon}$.