On Robust Output Based Finite-Time Control of LTI systems using HOSMs

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Abstract: Finite-Time stability of Linear Time Invariant systems with matched perturbations using dynamic output feedback is achieved under the assumptions of well-defined relative degree and a known bound of the perturbations. The approach is based on high order sliding modes, using global controllers and differentiator. A separation criteria that allows to detect the convergence of the differentiator and posterior gain adaptation is presented. Analysis of the performance under noise and sampling is presented.

Keywords: Dynamic output feedback, sliding-mode control.

1. INTRODUCTION

Motivation. Linear Time Invariant (LTI) systems are the class of systems where the most extensive research has been done, there exists a wide arsenal of tools to accomplish almost any desired control task. However, in real applications, to only consider a simple LTI model for control design automatically implies the need for robustness of the designed controller against unaccounted nonlinearities and perturbations. This way, it is clear that the simpler the model we have chosen the more robust the designed controller must be.

Sliding Mode (SM) Control is a technique that allows the design of robust, or better yet insensitive, controllers against matched perturbations Utkin (1992). In addition, it also features exact finite-time convergence Utkin (1992). This last property certainly is a useful one for controllers in Hybrid or Switching systems, because it can provide the exact convergence during dwell times, eliminating any error accumulation during successive switchings. Therefore it is enough to design a finite-time exact controller for each "operation mode", suppressing the hybrid nature of the problem and simplifying the control design.

Antecedents. Some results concerning finite-time stability of LTI systems can be found, for example, in Bhat and Bernstein (2000, 2005). However, in those results uncertainty is never considered and the whole state is assumed available for feedback. High Order SM (HOSM) controllers as finite-time universal controllers for uncertain SISO systems were introduced in Levant (2001). The design methodology is only based on the knowledge of the relative degree of the output and some bounds on the dynamic system. Results concerning the extension to multi-input case with Second Order SM, assuming full state information and bounded matched perturbations, can be found on Bartolini et al. (2000) where the authors obtain asymptotic stability of the origin. In Edwards et al. (2008), an output-based Second Order SM controller is presented for MIMO LTI systems with matched perturbations satisfying

- bounded-input bounded-state for both perturbations and control input,
- the perturbation and its first derivative are bounded by a known constant,
- well-defined relative degree,
- stable invariant zeroes.

Using a step-by-step Second Order SM algorithm for state observation, the authors obtain asymptotic stability of the origin. In spite of its recent theoretic development, HOSM controllers have found numerous applications, e.g. see Kunusch et al. (2009); Brambilla et al. (2008); Shtessel et al. (2007); Pisano and Usai (2004) for recent ones.

Main Contributions. We make use of global HOSM controllers, recently introduced in Levant and Michael (2008), to obtain global finite-time convergent output controllers for LTI systems with well-defined relative degree, considering a wider class of matched perturbations. The use of arbitrary order HOSM controllers allow us to improve the precision of the implemented controllers with respect to sampling and noise. A separation criteria to detect the convergence of the differentiator is presented allowing to turn-on the controller once all the estimation of the derivatives have converged. This same criteria is used to adapt the differentiator gain to obtain globality and better performance.

2. PROBLEM FORMULATION

Consider the system

$$\begin{aligned} \dot{x} &= Ax + B[u + w(t)] \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $w \in \mathbb{R}^m$ are the state, control input, measured output, unknown input (perturbation) signals, respectively. We assume that only the output $y(t)$ is available for feedback and that the unknown input belong to the class $\|w(t)\| \le W^+$, with W^+ a known constant. For the sake of clarity on the exposition we will assume that the system has "well-defined relative degree", to be introduced later.

3. DIFFERENTIATOR ANALYSIS

3.1 Differentiator properties

Throughout this paper we will assume that all the required derivatives are available in real time for feedback by means of real-time exact robust differentiators Levant (2003). Let $f(t) \in \mathbb{R}^1$ be a function to be differentiated. Then the k-th order differentiator takes the form

$$\begin{aligned}
\dot{z}_0 &= v_0 = -\lambda_k L^{\frac{1}{k+1}} |z_0 - f|^{\frac{k}{k+1}} \operatorname{sign}(z_0 - f) + z_1, \\
\dot{z}_1 &= v_1 = -\lambda_{k-1} L^{\frac{1}{k}} |z_1 - v_0|^{\frac{k-1}{k}} \operatorname{sign}(z_1 - v_0) + z_2, \\
&\vdots \\
\dot{z}_{k-1} &= v_{k-1} = -\lambda_1 L^{\frac{1}{2}} |z_{k-1} - v_{k-2}|^{\frac{1}{2}} \operatorname{sign}(z_{k-1} - v_{k-2}) + \\
&\quad + z_k, \\
\dot{z}_k &= -\lambda_0 L \operatorname{sign}(z_k - v_{k-1}).
\end{aligned} \tag{2}$$

where z_i is the estimation of the true signal $f^{(i)}(t)$. The differentiator ensures finite-time exact estimation in ideal conditions (no noise nor sampling) provided that L is an a-priory known upper bound for $|f^{(k+1)}|$ and the parametric sequence $\{\lambda_i\}$, $i = 0, 1, \ldots, k$ is chosen appropriately. In particular, the parameters $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$ are enough for the construction of differentiators up to the 5-th order. However, since the exact estimates required to construct the control signal are only available after a finite time, the more rational solution is first to wait until the finite-time exact estimate of the derivatives is ready and then to turn on the controller. But, when has the estimation of the HOSM differentiator converged? This question is answered by the following theorem

Theorem 1. Consider the HOSM differentiator (2) of order k with $f(t) \in \mathbb{R}^1$ is the signal to be differentiated. Assume that $\{\lambda_0, \ldots, \lambda_k\}$ were properly chosen and the differentiator provides for the finite-time estimation of the derivatives in ideal conditions. Let

$$f(t) = f_0(t) + \eta(t), \quad |f_0^{(k+1)}(t)| < L, \quad |\eta(t)| \le \varepsilon,$$

where $f_0(t)$ is an unknown basic signal and $\eta(t)$ is a Lebesgue-measurable noise. Suppose also that $f(t)$ is sampled with the time step $\tau > 0$, and $\varepsilon \le k_\varepsilon L \xi^{k+1}$, $\tau \le k_\tau \xi$, with ξ, k_ε and k_τ some positive constants. Then there exists positive constants $\gamma_0, \gamma_1, \ldots, \gamma_k$ and γ_t such that that if the inequality

$$|z_0 - f_0(t)| \le \gamma_0 L \xi^{k+1} \tag{3}$$

holds during the time interval $\gamma_t \xi$ then also

$$|z_i - f_0^{(i)}(t)| \le \gamma_i L \xi^{k-i+1}, \ i = 1, 2, \ldots k \tag{4}$$

hold, and moreover, starting from that moment inequalities (3), (4) are kept.

Remark. Notice that in any case the accuracies (3) and (4) are obtained after a finite transient time independently of ξ. In particular, exact estimations are obtained in the limit case $\xi = 0$, i.e. no noise and continuous sampling, Levant (2003).

Proof. Denote $\sigma_i := (z_i - f_0^{(i)})/L$, $\boldsymbol{\sigma} := [\sigma_0, \cdots, \sigma_k]^T$. Subtracting $f_0^{(i+1)}$ from the both sides of the equation on z_i of (2) and dividing by L, obtain in example for the first equation

$$\begin{aligned}
\dot{z}_0 - \dot{f}_0 = &-\lambda_k L^{\frac{1}{k+1}} |z_0 - f_0 - \eta|^{\frac{k}{k+1}} \operatorname{sign}(z_0 - f_0 - \eta) + \\
&+ z_1 - \dot{f}_0
\end{aligned}$$

since $z_0 - f_0 = L\sigma_0$ and $z_1 - \dot{f}_0 = L\sigma_1$, then

$$L\dot{\sigma}_0 = -\lambda_k L^{\frac{1}{k+1}} L^{\frac{k}{k+1}} |\sigma_0 - \eta/L|^{\frac{k}{k+1}} \operatorname{sign}(\sigma_0 - \eta/L) + L\sigma_1$$

and repeating the same procedure, obtain the differential inclusion

$$\begin{aligned}
\dot{\sigma}_0 &= -\lambda_k |\sigma_0 - \eta(t)/L|^{\frac{k}{k+1}} \operatorname{sign}(\sigma_0 - \eta(t)/L) + \sigma_1, \\
\dot{\sigma}_1 &= -\lambda_{k-1} |\sigma_1 - \dot{\sigma}_0|^{\frac{k-1}{k}} \operatorname{sign}(\sigma_1 - \dot{\sigma}_0) + \sigma_2, \\
&\cdots \\
\dot{\sigma}_{k-1} &= -\lambda_1 |\sigma_{k-1} - \dot{\sigma}_{k-2}|^{\frac{1}{2}} \operatorname{sign}(\sigma_{k-1} - \dot{\sigma}_{k-2}) + \sigma_k, \\
\dot{\sigma}_k &\in -\lambda_0 \operatorname{sign}(\sigma_k - \dot{\sigma}_{k-1}\ ) + [-1, 1].
\end{aligned} \tag{5}$$

where the last line of the inclusion is obtained due to $\dot{z}_k - f_0^{(k+1)} = \dot{\sigma}_k L$ and $f_0^{(k+1)} \in [-L, L]$.

The derivatives on the right-hand side of the last equations can be excluded in the following way. For the first equation $\eta/L \in [-k_\varepsilon \xi^{k+1}, k_\varepsilon \xi^{k+1}]$. From the first equation we can obtain that

$$|\sigma_1 - \dot{\sigma}_0| = \lambda_k |\sigma_0 - \eta/L|^{\frac{k}{k+1}}$$

so $\lambda_{k-1} |\sigma_1 - \dot{\sigma}_0|^{\frac{k-1}{k}} \le \lambda_k \lambda_{k-1} |\sigma_1 + [,]|^{\frac{k-1}{k+1}}$. Reasoning in the same way, we can obtain that

$$\lambda_{k-j} |\sigma_j - \dot{\sigma}_{j-1}|^{\frac{k-j}{k}} \le \tilde{\lambda}_{k-j} |\sigma_1 + [,]|^{\frac{k-j}{k+1}}$$

and result in the following non-recursive form

$$\begin{aligned}
\dot{\sigma}_0 &\in -\tilde{\lambda}_k \left|\sigma_0 + [-k_\varepsilon \xi^{k+1}, k_\varepsilon \xi^{k+1}]\right|^{\frac{k}{k+1}} \\
&\quad \times \operatorname{sign}(\sigma_0 + [-k_\varepsilon \xi^{k+1}, k_\varepsilon \xi^{k+1}]) + \sigma_1, \\
\dot{\sigma}_1 &\in -\tilde{\lambda}_{k-1} \left|\sigma_0 + [-k_\varepsilon \xi^{k+1}, k_\varepsilon \xi^{k+1}]\right|^{\frac{k-1}{k+1}} \\
&\quad \times \operatorname{sign}(\sigma_0 + [-k_\varepsilon \xi^{k+1}, k_\varepsilon \xi^{k+1}]) + \sigma_2, \\
&\cdots \\
\dot{\sigma}_{k-1} &\in -\tilde{\lambda}_1 \left|\sigma_0 + [-k_\varepsilon \xi^{k+1}, k_\varepsilon \xi^{k+1}]\right|^{\frac{1}{k+1}} \\
&\quad \times \operatorname{sign}(\sigma_0 + [-k_\varepsilon \xi^{k+1}, k_\varepsilon \xi^{k+1}]) + \sigma_k, \\
\dot{\sigma}_k &\in -\tilde{\lambda}_0 \operatorname{sign}(\sigma_0 + [-k_\varepsilon \xi^{k+1}, k_\varepsilon \xi^{k+1}])\ + [-1, 1].
\end{aligned} \tag{6}$$

The right hand side of (6) is minimally enlarged in order to provide for the convexity and upper-semicontinuity of the obtained differential inclusion Filippov (1960). The sampling of f corresponds to the time varying delay of the right-hand side not exceeding $k_\tau \xi$. Denoting (6) by $\dot{\boldsymbol{\sigma}} \in \Sigma(\boldsymbol{\sigma}(t), \xi)$, obtain that the system with sampling corresponds to

$$\dot{\boldsymbol{\sigma}} \in \Sigma(\boldsymbol{\sigma}(t - [0, k_\tau \xi]), \xi) \tag{7}$$

Now we make use of the following lemma, proved in the appendix,