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ORIGINAL ARTICLE

A new wavelet multigrid method for the numerical solution of elliptic type differential equations

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Abstract In this paper, we present a new wavelet multigrid method for the numerical solution of elliptic type differential equations based on Daubechies (db4) high pass and low pass filter coefficients with modified intergrid operators. The proposed method is the robust technique for faster convergence with less computational cost which is justified through the error analysis and condition number of a system in comparison with integrated-RBF technique based on Galerkin formulation (Mai-Duy and Tran-Cong, 2009) and finite difference method. Some of the illustrative problems are presented to demonstrate the attractiveness of the proposed technique.

1. Introduction

The mathematical modeling of engineering problems usually leads to sets of differential equations and their boundary conditions. To pursue solutions of differential equations, for most of the cases, it is necessary to employ discretization methods to reduce the differential equations into system of algebraic equations. System of algebraic equations is related to many problems arising in science and engineering, as well as with applications of mathematics to the social sciences and the quantitative study of business and economic problems. Direct methods are used to solve a linear system of N equations with N unknowns. Direct methods are theoretically producing the exact solution to the system in a finite number of steps. In practice, of course, the solution obtained will be polluted by the round-off error that is involved with the arithmetic being used. To minimize such round-off error iterative methods are infrequently used for solving linear systems. Since the time required for sufficient accuracy exceeds that required for direct methods. For large systems with a high quantity of 0 entries, however, these methods are efficient in terms of both computer storage and computation cost. This type of system stands up frequently in science and engineering problems in the numerical solution of elliptic type of problems. The multigrid method is largely applicable in increasing the efficiency of iterative methods used to solve large system of algebraic equations [2].

The multigrid (MG) method is a well-founded numerical method for solving sparse linear system of equations approximating differential equations. In the historical three decades the development of effective iterative solvers for systems of algebraic equations has been a significant research topic in numerical analysis and computational science and engineering. Nowadays it is recognized that multigrid iterative solvers are highly efficient for elliptic boundary value problems and often have optimal complexity. For a detailed treatment of multigrid methods we refer to Hackbusch [3]. An introduction of multigrid methods is found in Wesseling, Briggs and Trottenberg

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et al. [4–6]. Griebel and Knapek [7] used matrix-dependent interpolations, where the coarse grid operator is determined to be a Schur complement using a Galerkin approach. Ghaffar et al. [8–10] used the multigrid method for the solution of Helmholtz equation. However, when met by certain problems, the standard multigrid procedure converges slowly with larger computational time, whereas wavelet multigrid based methods reduce system of equations into well-conditioned with faster convergence in lesser computational cost [11].

The mathematical theory of wavelets is more than 25 years, yet already wavelets have become an important tool in many areas, such as image processing and time series analysis. In recent years, wavelet analysis is fast extensive kindness in the numerical solution of elliptic type of problems. The smooth orthogonal basis is obtained by the dilation and shift of a single special function, called "mother wavelet". Recently, many authors (De Leon [11] and Bujurke et al. [12]) have worked on wavelet multigrid methods. These methods use a choice of the filter operators obtained from wavelets to define the prolongation and restriction operators. Avudainayagam and Vani [13] used wavelet-based interpolation and restriction operators for their multigrid approaches, and Vasilyev and Kevlahan [14] applied a wavelet-collocation-based multigrid method for elliptic problems. Shiralashetti et al. [15] proposed the new wavelet based Full-approximation scheme for the numerical solution of non-linear elliptic partial differential equations. This paper outspreads the new approach of a new wavelet multigrid method (NWMG) to solve elliptic differential equations in which we obtained well-conditioned system by defining the condition number in comparison with [1]. Thus, the proposed method can be applied to a wide range of science and engineering problems.

The organization of this paper is as follows. In Section 2, Daubechies wavelets and wavelet multigrid operators are given. The method of solution is discussed in Section 3. Section 4 presents numerical examples and results. Finally, conclusions of the proposed technique are discussed in Section 5.

2. Daubechies wavelets

A major problem in the growth of wavelets during the 1980s was the search for a multiresolution analysis where the scaling function was compactly supported and continuous. As already we know that, the Haar multiresolution analysis is generated by a compactly supported scaling function but it is not continuous. The B-splines are continuous and compactly supported but fail to form an orthonormal basis. A family of multiresolution analysis was generated by scaling functions, which are both compactly supported and continuous. This multiresolution analysis was first constructed by Daubechies [16,17], that created great eagerness among mathematicians' and scientists' performance research in the area of wavelets.

2.1. Wavelet multigrid (WMG) operators

The matrix formulation of the discrete signals and discrete wavelet transforms (DWT) plays an important part in the wavelet method. This is highly expedient and informative, particularly for the numerical computations. As we already know about the DWT matrix and its applications in the wavelet method, it is given in [9] as,

$$W_1 = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & 0 & 0 & \dots & & 0 & 0 \\ g_0 & g_1 & g_2 & g_3 & 0 & 0 & \dots & & 0 & 0 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & \dots & & 0 & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & g_3 & \dots & & 0 & 0 \\ \vdots & & \ddots & & \dots & & \dots & & 0 & 0 \\ h_2 & h_3 & 0 & 0 & \dots & & \dots & 0 & h_0 & h_1 \\ g_2 & g_3 & 0 & 0 & \dots & & \dots & 0 & g_0 & g_1 \end{pmatrix}_{2^J \times 2^J}$$

where $h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$ are low pass filter coefficients and $g_0 = \frac{1-\sqrt{3}}{4\sqrt{2}}$, $g_1 = -\frac{3-\sqrt{3}}{4\sqrt{2}}$, $g_2 = \frac{3+\sqrt{3}}{4\sqrt{2}}$, $g_3 = -\frac{1+\sqrt{3}}{4\sqrt{2}}$ are the high pass filter coefficients.

Using this matrix authors used restriction and prolongation operators W and W^T respectively given in Section 3.2, alike to multigrid operators.

2.2. New wavelet multigrid (NWMG) operators

Here, we developed modified DWT matrix from DWT matrix in which we have added rows and columns consecutively with diagonal element as 1, which is built as,

$$W_2 = \begin{pmatrix} h_0 & 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & 0 & \dots & & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & & & & \dots & 0 & 0 \\ g_0 & 0 & g_1 & 0 & g_2 & 0 & g_3 & 0 & 0 & \dots & & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & & & \dots & 0 & 0 \\ \vdots & & \ddots & & \dots & & \dots & & & & & 0 & 0 \\ g_2 & 0 & g_3 & 0 & \dots & & \dots & 0 & & g_0 & 0 & g_1 & 0 \\ 0 & 0 & \dots & & \dots & & & \dots & & \dots & & 0 & 1 \end{pmatrix}_{2^J \times 2^J}$$

Using W_2 matrix, we developed restriction and prolongation operators WP and WP^T respectively alike to wavelet multigrid operators given in Section 3.3.

3. Method of solutions

Consider the elliptic differential equation of the type one and two dimensional problems, after discretizing the differential equation through the finite difference method (FDM), we get system of algebraic equations. Then it can be written in the matrix form as

$$Au = b \tag{3.1}$$

where A is $2^J \times 2^J$ coefficient matrix, b is $2^J \times 1$ matrix and u is $2^J \times 1$ matrix to be determined, where J is the maximum level of resolution. By solving Eq. (3.1) through the iterative method, we get the approximate solution v of u. That is, $u = e + v \Rightarrow v = u - e$, where e is ($2^J \times 1$ matrix) error to be determined. In numerical analysis, the approximate solution contains some error. Hence, there are many known approaches to minimize the error. Some of them are Multigrid (MG), Wavelet multigrid (WMG) and new wavelet multigrid (NWMG) Methods. Now we are deliberating about the method of solution of these mentioned methods as below.

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