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ORIGINAL ARTICLE

A new analytical approach for solving quadratic nonlinear oscillators

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Abstract In this paper, a new analytical approach based on harmonic balance method (HBM) is presented to obtain the approximate periods and the corresponding periodic solutions of quadratic nonlinear oscillators. The result obtained in new approach has been compared with that obtained by other existing method. The present method gives not only better result than other existing result but also gives very close to the corresponding numerical result (considered to be the exact result). Moreover, the method is simple and straightforward.

1. Introduction

The nonlinear problem often arises in exact modeling of phenomena in physical science, mechanical structures, nonlinear circuits, chemical oscillation and other engineering research and study of them is of interest to many researchers. For example, the eardrum is the best modeled by quadratic nonlinear oscillator [1]. Nowadays, several analytical methods such as homotopy perturbation [2], harmonic balance [3], residue harmonic balance [4], global residue harmonic balance [5], Hamiltonian [6], homotopy analysis [7], max-min [8], coupling of homotopy-variational [9], iterative homotopy harmonic balance method [10], Fourier series solutions with finite harmonic terms [11], and amplitude-frequency formulation [12] have been developed for solving strongly nonlinear oscillators. Earlier the classical perturbation methods [13–16] were used to solve weakly nonlinear problems. Recently, Hu [17] has used HBM to determine an approximate solution of a quadratic nonlinear oscillator, $\ddot{x} + x + \varepsilon x^2 = 0$; but the method is not a simple one. Hu [17] has obtained two separate harmonic balance solutions respectively for two regions $x > 0$ and $x < 0$. The solution is continuous, but the derivative does not exist when it cuts the axis. In the present article, a new analytical approximate technique based on HBM is presented to obtain the approximate solution of quadratic nonlinear oscillators. Here we obtain one trial solution and the solution is continuous and differentiable everywhere. The results are compared with those obtained by Hu [17] (see Appendix A).

2. Formulation and solution method

Consider a nonlinear differential equation

$$\ddot{x} + x = -\varepsilon f(x, \dot{x}), \quad x(0) = a, \dot{x}(0) = 0 \tag{1}$$

where $f(x, \dot{x})$ is a nonlinear function such that $f(-x, -\dot{x}) = f(x, \dot{x})$.

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Let us consider,

$$x(t) = a(c + \rho\cos\varphi + u\cos 2\varphi + v\cos 3\varphi + \ldots) \tag{2}$$

be a solution of (1), where a, c, ρ are constants, $\varphi = \omega t$ and $\omega = \frac{2\pi}{T}$ is a frequency of nonlinear oscillation, here T is a period. If $\rho = 1 - c - u - v - \cdots$ and the initial phase $\varphi_0 = 0$, solution Eq. (2) readily satisfies the initial conditions $x(0) = a, \dot{x}(0) = 0$. Substituting Eq. (2) in Eq. (1) and expanding $f(x, \dot{x})$ in a Fourier series, it turns to an algebraic identity

$$a\left[\rho(1 - c - \dot{\varphi}^2)\cos\varphi + u(1 - 4\dot{\varphi}^2)\cos 2\varphi \cdots\right]$$
$$= -\varepsilon[F_1(a, c, u, \cdots)\cos\varphi + F_2(a, c, u, \cdots)\cos 2\varphi \cdots]. \tag{3}$$

Equating the coefficients of equal harmonics of Eq. (3), the following nonlinear algebraic equations are found:

$$\rho(1 - c - \dot{\varphi}^2) = -\varepsilon F_1, u(1 - 4\dot{\varphi}^2) = -\varepsilon F_2,$$
$$v(1 - 9\dot{\varphi}^2) = -\varepsilon F_3, \cdots \tag{4}$$

with the help of second equation, $\dot{\varphi}$ is eliminated from all the rest of Eq. (4). Thus Eq. (4) takes the following form

$$\rho\dot{\varphi}^2 = \rho + \varepsilon F_1 - \rho c, 3\rho u = \rho\varepsilon F_2 - 4u\varepsilon F_1 + 4\rho uc,$$
$$8\rho v = \rho\varepsilon F_3 - 9v\varepsilon F_2 + 9\rho vc, \cdots \tag{5}$$

using $\rho = 1 - c - u - v - \cdots$ and simplifying, second, third-equations of Eq. (5) takes the following nonlinear algebraic equations

$$G_1(a, \varepsilon, c, u, v, \cdots) = 0, \; G_2(a, \varepsilon, c, u, v, \cdots) = 0, \cdots. \tag{6}$$

These types of algebraic equations have been solved by the power series method introducing a small parameter (see [18,19] for details) which provides desired results.

3. Example

Consider the quadratic nonlinear equation in the following form

$$\ddot{x} + x + \varepsilon x^2 = 0, \; x(0) = a, \; \dot{x}(0) = 0. \tag{7}$$

The third-order approximate solution is chosen in the following form

$$x = a(c + \rho\cos\varphi + u\cos 2\varphi + v\cos 3\varphi) \tag{8}$$

where

$\rho = 1 - c - u - v$ and $\varphi = \omega t$.

Substituting Eq. (8) into Eq. (7) and expanding in a Fourier series and equating the constant terms and the coefficients of $\cos\varphi$, $\cos 2\varphi$ and $\cos 3\varphi$ respectively, we obtained the following equations as

$$c + \frac{1}{2}a\varepsilon\left(2c^2 + u^2 + v^2 + \rho^2\right) = 0 \tag{9}$$

$$auv\varepsilon + \rho(1 + 2ac\varepsilon + au\varepsilon) - \rho\omega^2 = 0 \tag{10}$$

$$u + a\varepsilon\left(2cu + v\rho + \frac{1}{2}\rho^2\right) - 4u\omega^2 = 0 \tag{11}$$

$$v + a\varepsilon(2cv + u\rho) - 9v\omega^2 = 0. \tag{12}$$

By elimination of ω^2 from Eqs. (10)–(12), we obtained the following equations as

$$-4au^2v\varepsilon - \rho\left(3u + a\varepsilon\left(6cu + 4u^2 - v\rho - \frac{1}{2}\rho^2\right)\right) = 0 \tag{13}$$

$$-9auv^2\varepsilon - \rho(8v + a\varepsilon(16cv + 9uv - u\rho)) = 0. \tag{14}$$

Neglecting the higher order terms more than two such as u^2v and uv^2 from Eqs. (13) and (14) and also dividing Eqs. (13) and (14) by $-\rho$ we obtain as follows

$$3u + a\varepsilon\left(6cu + 4u^2 - v\rho - \frac{1}{2}\rho^2\right) = 0 \tag{15}$$

$$8v + a\varepsilon(16cv + 9uv - u\rho) = 0. \tag{16}$$

Substituting $\rho = 1 - c - u - v$ in Eqs. (15) and (16) then we obtain

$$a\varepsilon(1 + c^2) - 6u - 2a\varepsilon(c + u + 5cu + 7u^2 + v^2) = 0 \tag{17}$$

$$-au\varepsilon(-1 + c) - 8v - a\varepsilon(u^2 + 16cv + 10uv) = 0. \tag{18}$$

Here the coefficient of u of Eq. (17) is 6 and $a\varepsilon \leqslant 1/2$, $\varepsilon > 0$. On the other hand, the coefficient of v of Eq. (18) is 8 and v fully depends on u. Therefore, Eqs. (17) and (18) can be solved in power series by choosing a small parameter $\lambda = a\varepsilon/6$. Thus we obtain

$$u = (-1 + c)^2\left(\lambda - 2(1 + 5c)\lambda^2 + 3(-1 + 18c + 31c^2)\lambda^3\right.$$
$$\left.- 2\left(-17 - 3c + 489c^2 + 395c^3\right)\lambda^4\right) \tag{19}$$

$$v = (-1 + c)^3\left(-\frac{3}{4}\lambda^2 + \frac{9}{4}(1 + 7c)\lambda^3 - \frac{3}{8}(-13 + 242c + 635c^2)\lambda^4\right). \tag{20}$$

Now Eq. (9) can be solved for c by substituting the values of u and v from Eqs. (19) and (20). But we use another equation to find the value of c. When $\varphi \to \pi$, x (presented in

Table 1 Comparison of approximate periods with the corresponding exact period and Hu [17] for $\varepsilon = 1$.

a	Exact	Hu [17] Er(%)	Present study Er(%)
0.10	6.3116	6.3112 0.006	6.3116 0.000
0.20	6.4114	6.4095 0.029	6.4114 0.000
0.30	6.6294	6.6226 0.103	6.6290 0.006
0.40	7.1246	7.0962 0.399	7.1206 0.056
0.45	7.7065	7.6277 1.023	7.6939 0.163
0.46	7.9052	7.8014 1.313	7.8884 0.213
0.47	8.1672	8.0233 1.762	8.1466 0.252
0.48	8.5452	8.3278 2.544	8.5250 0.236
0.49	9.2080	8.8118 4.303	9.2202 0.132

Where Er(%) denotes the absolute percentage error.

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