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ORIGINAL ARTICLE

A moving least squares meshless method for solving the generalized Kuramoto-Sivashinsky equation

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Abstract We use a moving least squares meshless method to solve the nonlinear Kuramoto-Sivashinsky equation. The accuracy of the method is demonstrated by three test problems for which the numerical results are found to be in excellent agreement with analytical solution.

1. Introduction

The Kuramoto-Sivashinsky (KS) [1,2] equation is a nonlinear fourth order partial differential equation that has been proposed in the seventies for describing turbulence in reactive systems and diffusive instabilities in laminar flame fronts. Depending on the associated parameters this equation can be seen as an example of complex spatiotemporal dynamics leading to chaotic behavior. It has therefore been the subject of extensive analytical and numerical studies. Finite difference, finite volume and finite element methods have been used for the spatial discretization [3–7]. In order to use a Lattice Boltzmann method Lai and Ma [8] have proposed to construct a five velocity lattice Boltzmann model by introducing an amending function. Their results were found to be very accurate. A meshfree method using radial basis function (RBF) for the space discretization has also been proposed recently [9]. In this work, we introduce the moving least squares meshless method to solve the KS equation. This approach has already been used to solve several problems in heat transfer and related fluid flow problems ranging from natural and forced convection to radiative transfer in participating media [10–17]. In all these works second order in space meshless discretization has been used successfully. In [18] we have considered higher order meshless approximations to solve second order diffusion and transport-diffusion type equations. This meshless discretization technique can be found in the literature under multiple denominations as in [19,20]. Under several other formulations, the meshless approaches are still the subject of numerous developments [21–26]. In the following sections, the fourth order meshless approximation method is first described. The same three numerical cases studied by Lai and Ma [8] are then considered. It is found that the meshless results are in excellent agreement with the exact solutions.

2. Fourth order in space meshless method

Let us consider the following Taylor development at order 4 around a point M of coordinate x for a neighbor point M_i of coordinate x_i of the space discretization:

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$$\varphi_i^*(x_i) = \varphi + (x_i - x)\frac{d\varphi}{dx} + \frac{(x_i - x)^2}{2!}\frac{d^2\varphi}{dx^2} + \frac{(x_i - x)^3}{3!}\frac{d^3\varphi}{dx^3} + \frac{(x_i - x)^4}{4!}\frac{d^4\varphi}{dx^4} + O(\Delta x^4) \quad (1)$$

This can be written in more compact form as follows:

$$\varphi_i^*(x_i) = \langle p(M_i, M)\rangle \cdot \langle \alpha_M\rangle^T \quad (2)$$

$\langle p(M_i, M)\rangle$ is the line vector of the generalized polynomial basis and $\langle \alpha_M\rangle^T$ the transpose vector of the generalized variables of the approximation which are the successive derivatives.

Hence we have the following:

$$\langle p(M_i, M)\rangle = \langle 1, (x_i - x), (x_i - x)^2, (x_i - x)^3, (x_i - x)^4\rangle \quad (3)$$

and

$$\langle \alpha_M\rangle^T = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle^T \quad (4)$$

If the discrete φ_i^* values of function φ are supposed to be known on n neighboring nodes M_i, one can relate the approximations of the successive derivatives to the discrete values φ_i^*, by minimizing the following quadrature error:

$$I(\alpha_M) = \sum_{i=1}^{N}\left\{\omega(M_i, M)\left[\varphi_i - \langle p(M_i, M)\rangle\langle \alpha_M\rangle^T\right]^2\right\} \quad (5)$$

where ω is a positive weight function of compact support centered at the considered point M and rapidly decaying. The support of this function will define the number of neighboring nodes used for the approximation.

One can now minimize the quadratic form by writing the following:

$$\frac{\partial I(\alpha_M)}{\partial \alpha_i} = 0 \quad \text{for } i = 0, \ldots 4 \quad (6)$$

This leads to the system:

$$\left[A^M\right] \cdot \begin{Bmatrix} \varphi \\ \frac{d\varphi}{dx} \\ \frac{d^2\varphi}{2!dx^2} \\ \frac{d^3\varphi}{3!dx^3} \\ \frac{d^4\varphi}{4!dx^4} \end{Bmatrix}^* = \langle B^M\rangle^T \quad (7)$$

where $[A^M]$ and $\langle B^M\rangle^{\mathrm{T}}$ are defined by the following: $\langle B^M\rangle^T$

$$[A^M] = \sum_{i=1}^{N}\omega(M_i, M)\langle p(M_i, M)\rangle^T\langle p(M_i, M)\rangle \quad (8)$$

$$\langle B^M\rangle^T = \sum_{i=1}^{N}\omega(M_i, M)\langle p(M_i, M)\rangle^T.\varphi_i \quad (9)$$

If matrix $[A^M]$ is not singular, the system (7) can be inverted:

$$\begin{Bmatrix} \varphi \\ \frac{d\varphi}{dx} \\ \frac{d^2\varphi}{2!dx^2} \\ \frac{d^3\varphi}{3!dx^3} \\ \frac{d^4\varphi}{4!dx^4} \end{Bmatrix}^* = \left[A^M\right]^{-1} \cdot \left\{\sum_{i=1}^{n}\omega(M_i, M) \cdot \langle p(M_i, M)\rangle^T \cdot \varphi_i\right\} \quad (10)$$

and the successive derivatives of the unknown function at point M are expressed in terms of the different values φ_i of the function in the chosen neighboring nodes.

If $\langle a_j\rangle$ represents the jth line of the inverse matrix $\left[A^M\right]^{-1}$, the derivatives now simply write the following:

$$\frac{\partial\varphi}{\partial x} = \sum_{M_i \in V^M}\omega(M_i, M)\langle a_1\rangle\langle p(M_i, M)\rangle^T.\varphi_i \quad (11)$$

$$\frac{\partial^2\varphi}{\partial x^2} = 2!\sum_{M_i \in V^M}\omega(M_i, M)\langle a_2\rangle\langle p(M_i, M)\rangle^T.\varphi_i \quad (12)$$

$$\frac{\partial^3\varphi}{\partial x^3} = 3!\sum_{M_i \in V^M}\omega(M_i, M)\langle a_3\rangle\langle p(M_i, M)\rangle^T.\varphi_i \quad (13)$$

$$\frac{\partial^4\varphi}{\partial x^4} = 4!\sum_{M_i \in V^M}\omega(M_i, M)\langle a_4\rangle\langle p(M_i, M)\rangle^T.\varphi_i \quad (14)$$

The present collocation meshless method uses the strong formulation of the equation to solve in the sense that at each calculation point, all the derivatives appearing in the equation are replaced by their approximations given by previous expressions (11)–(14) leading thus to an algebraic equation at the point and finally to a system of N algebraic equations if N calculation points are used in the spatial discretization. Boundary conditions are introduced to the algebraic system which is then solved once in a steady state problem or at each time step in an unsteady problem which is the case herein. For the time discretization we used a simple Euler implicit scheme although more accurate schemes could be used.

The weighting function can have several forms (triangular, Hanning, exponential...). In this work the following Gaussian function has been employed:

$$\omega(r) = \mathrm{Exp}\left(\ln(\varepsilon)\left(\frac{r}{S}\right)^2\right)$$

where $r = |MM_i|$ represents the distance between points M and his neighbors M_i and where S is the size of the function support. It is important to note that the weight function must be sufficiently large to enclose a number of nodes at least equal to the number of generalized variables. Finally, our previous works [10–17] have shown that ε value can be chosen in the range $[10^{-3}$–$10^{-9}]$, depending on the number of selected nodes. In this work, a constant value of 10^{-6} is chosen.

3. Application to the Kuramoto-Sivashinsky equation

We present in this section the results obtained by the previous meshless method when applied to the following KS equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0 \quad (15)$$

where α, β and γ are problem dependant constants. The three particular problems studied in [8] are used to test the present approach. As in [8] the global relative error (GRE) is introduced for testing the precision:

$$\mathrm{GRE} = \frac{\sum_i|u^*(x_i, t) - u(x_i, t)|}{\sum_i|u(x_i, t)|} \quad (16)$$

where u^* is the numerical solution and u the exact solution.

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