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ORIGINAL ARTICLE

Flow over an infinite plate of a viscous fluid with non-integer order derivative without singular kernel

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Abstract Exact general solutions for the dynamics of an incompressible viscous fluid with non-integer order derivative without singular kernel are established using the integral transforms. These solutions, which are organized in simple forms in terms of exponential and trigonometric functions, can be conveniently engaged to obtain known solutions from the literature. The control of the new non-integer order derivative on the velocity of the fluid moreover a comparative study with an older model, is analyzed for some flows with practical applications. The non-integer order derivative with non-singular kernel is more appropriate for handling mathematical calculations of obtained solutions. It is also more reliable for numerical computations.

1. Introduction

A discussion on the fractional calculus and its applications at the actual moment, is almost without sense. Fractional calculus is approximately as been around as the standard differential and integral calculus and a list of its applications is too long to be here included. However, it is important to emphasize the fact that fractional derivative generalizations of one-dimensional viscoelastic models have been found to be of great utility in modeling the response linear regime [1] and they are in agreement with the second principle of thermodynamics. Furthermore, as it results from the work of Makris et al. [2], a satisfactory agreement of experimental work was achieved when the non-integer order Maxwell model was used in place of the ordinary one. They also proved that the fractional model has a stronger memory of the recent past than the ordinary model.

During the last decades the fractional calculus has been extensively used and a lot of motion problems have been studied using it [3]. As usually, the governing equations analogous to motions of ordinary fluid models are modified by rehabilitating the integer order time derivatives by the formal left-hand Liouville or Riemann-Liouville differential operators [4–6]. However, these operators as well as the Caputo operator have some drawbacks. Their kernel is singular and the most results that have been recovered using them are expressed in complicated forms involving some generalized functions even for Newtonian fluids [7,8].

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Recently, Caputo and Fabrizio [9] provided a modern definition of non integer order derivative with smooth kernel that works both for temporal and spatial variables. Due to its advantage when the Laplace transform is employed to do problems with initial conditions, this derivative has been already used to solve different real problems [10,11]. A new fractional operator whose kernel is also non-singular was proposed by Atangana and Baleanu [12]. It is based on the Mittag-Leffler function and is useful in thermal science and material sciences. It is worth pointing out that both fractional derivatives, Caputo-Fabrizio and Atangana Baleanu, have all benefits of Riemann-Liouville and Caputo operators. In addition, their kernel is non-singular.

The aim of this note is to use the advantages of one of the modern definitions of non-integer order derivative to obtain exact general solutions for the flow of an incompressible fractional viscous fluid over an infinite plate that is moving in its plane or applies an arbitrary time-dependent tangential stress to the fluid. The obtained results are expressed in simpler forms involving exponential and trigonometric functions and can be conveniently engaged to recover the related solutions for ordinary fluids. Finally, for validation as well as for comparison, three motions with scientific value are considered to obtain as limiting case several results from the literature. Velocity profiles corresponding to two of them are graphically presented and the necessary time to attain the steady-state for oscillating motions is obtained for both models i.e. with Caputo-Fabrizio and Caputo fractional derivative operators.

2. Statement of the problem

Consider an infinite plate situated in the (x, y) plane of a fixed Cartesian coordinate system whose positive y-axis is in the upward direction, an incompressible viscous fluid is at rest over it at $t = 0$. For the time $t = 0^+$, the plate begins to displace in its plane with a time dependent velocity $Uf(t)$ along the x-axis or to apply a shear stress $Sg(t)$ to the fluid in the same direction, the motion is uniform (translation-invariant) in the x and z directions. Here U and S are constants while the non-dimensional functions $f(\dot{s})$ and $g(\dot{s})$ are piecewise continuous and $f(0) = g(0) = 0$. Due to the tangential stress the fluid is also moved and its velocity is of the form of

$$\mathbf{v} = \mathbf{v}(y,t) = (u(y,t), 0, 0). \tag{1}$$

For such motions, the continuity equation is identically verified while the motion and constitutive equations lead to the relevant partial differential equations

$$\frac{\partial \tau(y,t)}{\partial y} = \rho \frac{\partial u(y,t)}{\partial t}, \quad \tau(y,t) = \mu \frac{\partial u(y,t)}{\partial y}; \quad y,t > 0, \tag{2}$$

if the body forces as well as pressure gradient along the flow direction are neglected. Here ρ is the fluid density, μ is its viscosity and $\tau(y, t)$ is the non-zero shear stress.

Relevant initial and boundary conditions corresponding to the two different motions with velocity or shear stress on the boundary are

$$u(y,0) = 0, \quad y > 0; \quad u(0,t) = Uf(t), \quad t > 0, \tag{3}$$

respectively,

$$\tau(y,0) = 0, \quad y > 0; \quad \tau(0,t) = Sg(t), \quad t > 0. \tag{4}$$

Of course, the natural conditions at infinity, namely $u(y, t) \to 0$ and $\tau(y, t) \to 0$ as $y \to \infty$ must be satisfied.

Now to develop the solutions free from the geometry of flow management, we propose the following non-dimensional variables and functions:

$$y^* = \frac{U}{\nu} y, \quad t^* = \frac{U^2}{\nu} t, \quad u^* = \frac{u}{U}, \quad \tau^* = \frac{1}{\rho U^2}\tau,$$
$$f^*(t^*) = f\left(\frac{\nu}{U^2} t^*\right) \tag{5}$$

and

$$y^* = \frac{1}{\nu}\sqrt{\frac{S}{\rho}} y, \quad t^* = \frac{S}{\mu} t, \quad u^* = \sqrt{\frac{\rho}{S}} u, \quad \tau^* = \frac{\tau}{S},$$
$$g^*(t^*) = g\left(\frac{\mu}{S} t^*\right), \tag{6}$$

corresponding to the two different motion problems.

Substituting Eqs. (5) or (6) into Eqs. (2)–(4) and withdrawing the star notation, we find the following non-dimensional initial-boundary value problem:

$$\frac{\partial \tau(y,t)}{\partial y} = \frac{\partial u(y,t)}{\partial t}, \quad \tau(y,t) = \frac{\partial u(y,t)}{\partial y}; \quad y,t > 0, \tag{7}$$

$$u(y,0) = 0, \quad y > 0; \quad u(0,t) = f(t), \quad t > 0;$$
$$u(y,t) \to 0, \text{ as } y \to \infty, \tag{8}$$

respectively

$$\tau(y,0) = 0, \quad y > 0; \quad \tau(0,t) = g(t), \quad t > 0;$$
$$\tau(y,t) \to 0, \text{ as } y \to \infty. \tag{9}$$

Eliminating $\tau(y, t)$ or $u(y, t)$ between Eq. (7), we noticed that the velocity and the shear stress corresponding to both motions satisfy linear partial differential equations of the same style, namely

$$\frac{\partial u(y,t)}{\partial t} = \frac{\partial^2 u(y,t)}{\partial y^2} \quad \text{or} \quad \frac{\partial \tau(y,t)}{\partial t} = \frac{\partial^2 \tau(y,t)}{\partial y^2}; \quad y,t > 0. \tag{10}$$

Consequently, as the initial and boundary conditions corresponding to both the problems are also identical in form, it is sufficient to solve one of problems and then to use Eq. (7) in order to find the solution of the other problem.

The fractional models corresponding to the two problems are based on the non-integer order partial differential equations [7, Eq. (6.7.44)]:

$$D_t^\alpha u(y,t) = \frac{\partial^2 u(y,t)}{\partial y^2} \quad \text{or} \quad D_t^\alpha \tau(y,t) = \frac{\partial^2 \tau(y,t)}{\partial y^2}; \quad y,t > 0, \tag{11}$$

with the initial and boundary conditions given by Eqs. (8), respectively (9). Here, unlike the previous published papers, the Caputo-Fabrizio derivative operator of order α [9]

$$D_t^\alpha[h(t)] = \frac{1}{1-\alpha}\int_0^t h'(s)\exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] ds \text{ for } 0 \leqslant \alpha \leqslant 1, \tag{12}$$

will be used. We firstly solve the fractional differential Eq. $(11)_1$ with the conditions (8) and use the obtained results to develop the solution corresponding to the second problem.

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