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ORIGINAL ARTICLE

New version of Optimal Homotopy Asymptotic Method for the solution of nonlinear boundary value problems in finite and infinite intervals

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Abstract In this research work a new version of Optimal Homotopy Asymptotic Method is applied to solve nonlinear boundary value problems (BVPs) in finite and infinite intervals. It comprises of initial guess, auxiliary functions (containing unknown convergence controlling parameters) and a homotopy. The said method is applied to solve nonlinear Riccati equations and nonlinear BVP of order two for thin film flow of a third grade fluid on a moving belt. It is also used to solve nonlinear BVP of order three achieved by Mostafa et al. for Hydro-magnetic boundary layer and micro-polar fluid flow over a stretching surface embedded in a non-Darcian porous medium with radiation. The obtained results are compared with the existing results of Runge-Kutta (RK-4) and Optimal Homotopy Asymptotic Method (OHAM-1). The outcomes achieved by this method are in excellent concurrence with the exact solution and hence it is proved that this method is easy and effective.

1. Introduction

Various physical models in Engineering and Science can be formulated in terms of boundary value problems (BVPs). These are used in the mathematical modeling of different entities such as visco-elastic flows, hydrodynamic stability problems, non-Newtonian fluids, and convection of heat [1].

Nonlinear BVPs have numerous applications in almost every field of Science and Engineering. Applications of first and second order BVPs can be found in many books at undergraduate level. First order BVPs are applied in fluid dynamics e.g. in design of containers and funnels. It can be applied in heat conduction analysis like design of heat spreaders in micro electronics and it can also be used in combined heat conduction and convection e.g. design of heating and cooling chambers. Second-order BVPs have a variety of applications in Science and Engineering like the vibration of spring and

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an electric circuit. Alfven waves in rotating inhomogeneous plasmas, the shear deformation of a sandwich beams, draining and coating flows are modeled by third order problems [2–5].

Many attempts have been made to solve BVPs on finite and infinite intervals which do arise in Engineering and Applied Sciences. Numerical methods such as Shooting method [6], Runge Kutta method [7], Finite difference method [7,8], Finite element method [9], Radial basis function methods and methods based on wavelets have been so far helpful. Other approximate methods which are used are Adomian decomposition method [10–12] and Variation iteration method (VIM) [13,14].

Perturbation and Asymptotic techniques are also widely applied to achieve analytic approximations of nonlinear BVPs in science, finance and engineering. Unfortunately, perturbation and asymptotic techniques are dependent upon physical parameters (small/large) in general and thus are often valid only for weak nonlinear problems. Therefore, some analytic approximation methods are needed which are not dependent upon physical parameters at all and valid for nonlinear problems. A well-known method based upon the idea of homotopy and which is also independent of physical parameters is Homotopy Analysis Method (HAM) [15,16]. This idea of homotopy was introduced by Shijun Liao in his PhD thesis (1992) for the first time, which is a basic concept in topology and differential geometry. This method has been used by different researchers to achieve series solution of different linear and nonlinear problems [17–19].

Researchers have introduced many other methods based on HAM and Homotopy Perturbation Method (HPM) [20–24], for example, OHAM [25–29,31,32], Third alternative of OHAM [30] and Optimal Homotopy Perturbation Method (OHPM) [33–36] to get the approximate solution of the nonlinear BVPs. But it is still quite problematic and need new techniques for finding the approximate solutions.

Our purpose in this contribution is to achieve a new version of OHAM by taking help from the book [30] which produces more accurate and reliable results than OHAM. The goal is achieved here by using initial guess, auxiliary functions, auxiliary convergence controlling parameters, and a homotopy in a particular way to make Optimal Homotopy Asymptotic Method simple and effective. The new version of OHAM is represented by OHAM-2. Here, nonlinear Riccati equations [37], nonlinear BVP of order two for thin film flow of a third grade fluid on a moving belt [32], and nonlinear BVP of order three achieved by Mostafa et al. for Hydro-magnetic boundary layer micro-polar fluid flow over a stretching surface embedded in a non-Darcian porous medium with radiation [38,39] are solved to show the potential of the applied method. The achieved results are compared with exact solution, RK-4 solution, and with first version of OHAM (OHAM-1). Numerical results show that OHAM-2 is found the best in giving better and more accurate results. It consists of few steps and converges to almost exact solution. The applied method is simple in learning and easy to apply. This method has great potential to solve ordinary differential equations of any order. The same technique can also be extended to solve partial differential equations, Integro-differential equations and system of differential equations of physical phenomenon. Here math type and mathematica 7.0 are for calculations as well as numerical simulations.

2. Materials and methods

2.1. Fundamental concept of OHAM

This method was achieved by Marinca and Herisanu [25–27] in 2008 for the first time and was based on the concept of homotopy. The method is explained as follows:

Suppose the following boundary value problem:

$$\Upsilon(\mu(s)) + f(s) = \Phi(\mu(s)) + f(s) + \Psi(\mu(s)) = 0,$$
$$\beta\left(\mu(s), \frac{d\mu(s)}{ds}\right) = 0, \tag{1}$$

where $f(s)$ is a known function, p is an embedding parameter, β is a boundary operator, s is independent variable, and $\mu(s)$ is an undetermined function. Also Υ is a general operator, Φ is linear operator, and Ψ is nonlinear operator. In this method we define a homotopy: $\mathbf{H}(v(s,p,c_i)) : \mathbf{\Omega} \times [0,1] \to \mathbf{R}$ which satisfies

$$\begin{aligned}\mathbf{H}(v(s,p,c_i)) &= (1-p)\Phi(v(s,p,c_i) + f(s)) \\ &= H(s,p,c_i)\Phi(v(s,p,c_i) + f(s)) \\ &\quad + \Psi(v(s,p,c_i)).\end{aligned} \tag{2}$$

Here $s\epsilon\mathbf{R}$, Ω is the domain of interest, $H(s,p,c_i)$ is an auxiliary function which is nonzero for $p \neq 0$, $H(s,0,c_i) = 0$, and $v(s,p,c_i)$ is an undetermined function. Clearly, when $p = 0$ then:

$$v(s,0,c_i) = \mu_0(s,c_i) \tag{3}$$

and when $p = 1$ then

$$v(s,1,c_i) = \mu(s,c_i). \tag{4}$$

Therefore, the solution $v(s,p,c_i)$ changes from $\mu_0(s)$ to $\mu(s)$ as p changes from 0 to 1. Now the initial guess $\mu_0(s)$ is calculated from Eq. (2) for $p = 0$ and we have:

$$\Phi(\mu_0(s)) + f(s) = 0, \quad \beta\left(\mu_0(s), \frac{d\mu_0(s)}{ds}\right) = 0. \tag{5}$$

Now consider the auxiliary function $H(s,p,c_i)$ as follows:

$$H(s,p,c_i) = pH_1(s,c_i) + p^2H_2(s,c_i) + \cdots, \tag{6}$$

where the auxiliary functions $H_i(s,c_j)$, $i = 1,2,\ldots$ depend upon s and also on c_j, $j = 1,2,\ldots,s$.

Expand, $v(s,p,c_i)$ in Taylors series about p as follows:

$$v(s,p,c_i) = \mu_0(s) + \sum_{k=1}^{\infty}\mu_k(s,c_1,c_2,\ldots,c_k)p^k. \tag{7}$$

Now put Eq. (7) in Eq. (2) and compare the coefficients of the same powers of p to achieve linear equations as follows

Zeroth order problem:

$$\Phi(\mu_0(s)) + f(s) = 0, \quad \beta\left(\mu_0(s), \frac{d\mu_0(s)}{ds}\right) = 0. \tag{8}$$

First order problem:

$$\Phi(\mu_1(s)) + f(s) = c_1\Psi_0(\mu_0(s)), \quad \beta\left(\mu_1(s), \frac{d\mu_1(s)}{ds}\right) = 0. \tag{9}$$

The required governing equations for $\mu_k(s)$ are given by: