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Model reduction, data-based and advanced discretization in computational mechanics

Wavelet-based multiscale proper generalized decomposition

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ABSTRACT

Separated representations at the heart of Proper Generalized Decomposition are constructed incrementally by minimizing the problem residual. However, the modes involved in the resulting decomposition do not exhibit a clear multi-scale character. In order to recover a multi-scale description of the solution within a separated representation framework, we study the use of wavelets for approximating the functions involved in the separated representation of the solution. We will prove that such an approach allows separating the different scales as well as taking profit from its multi-resolution behavior for defining adaptive strategies.

1. Introduction

Model Order Reduction – MOR – techniques allow nowadays solving, under real-time constraints, complex models. Intense research activities allowed reaching at present a certain maturity in the domain of model order reduction. Among the numerous references, the interested reader can refer to some review papers and books [1–4], covering three major MOR technologies: POD (Proper Orthogonal Decomposition), RB (Reduced Basis), and PGD (Proper Generalized Decomposition).

Proper Orthogonal Decomposition (POD) is a general technique for extracting the most significant characteristics of a system's behavior and representing them in a set of "POD basis vectors". These basis vectors then provide an efficient (typically low-dimensional) representation of the key system behavior, which proves useful in a variety of ways. The most common use is to project the system-governing equations onto the reduced-order subspace defined by the POD basis vectors. This yields an explicit POD reduced model that can be solved in place of the original system. The POD basis can also provide a low-dimensional description in which to perform parametric interpolation, infill missing or "gappy" data, perform model adaptation, or define hyper-reduction procedures [5]. There is an extensive literature and POD has seen broad application across fields. Some review of POD and its applications can be found in [6,7].

Another family of model reduction techniques lies in the use of Reduced Basis constructed by combining a greedy algorithm and "a posteriori" error indicators. As for the POD, the Reduced Basis method requires some amount of offline work, but then the reduced basis model can be used online for solving different models with control of the solution

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accuracy, because of the availability of error bounds. When the error is unacceptably high, the reduced basis can be enriched by invoking a greedy adaption strategy [8,9].

Separated representations, at the heart of the so-called Proper Generalized Decomposition methods, are considered when solving at-hand partial differential equations by employing procedures based on the separation of variables. Then they were considered in quantum chemistry for approximating multidimensional quantum wave-function. In the 1980s, Pierre Ladevèze proposed the use of space-time separated representations of transient solutions involved in strongly nonlinear models, defining a non-incremental integration procedure [10,11]. Later, separated representations were employed for solving multidimensional models suffering the so-called curse of dimensionality [12,13], as well as in the context of stochastic modeling [14]. Then, they were extended to the separation of space coordinates, making possible the solution to models defined in degenerated domains, e.g., plate and shells [15], as well as for addressing parametric models where model parameters were considered as model extra-coordinates, making possible the offline calculation of the parametric solution, which can be viewed as a metamodel or a computational vademecum to be used online for real-time simulation, optimization, inverse analysis, and simulation-based control [2,16].

1.1. Separated representations

Within the PGD framework, four kinds of separated representations have been widely considered.

(i) *Space-time separated representations* that allowed the construction of efficient incremental and non-incremental integrators.
Within the standard finite element method, a space-time solution $u(\mathbf{x},t)$, $\mathbf{x}\in\Omega\subset\mathbb{R}^3$ and $t\in\mathcal{I}\subset\mathbb{R}$, of a transient problem is approximated from

$$u(\mathbf{x},t)\approx\sum_{i=1}^{\mathtt{M}}u(\mathbf{x}_i,t)N_i(\mathbf{x}) \tag{1}$$

where $\mathtt{M}$ is the number of nodes employed for interpolating the unknown field, located at positions $\mathbf{x}_i$, and $N_i(\mathbf{x})$ the so-called shape functions. Because of the interpolative property of the shape functions, the approximation coefficients correspond to the nodal value of the approximated field, $u(\mathbf{x}_i,t)$. Thus, in general, when solving a nonlinear problem, at least a linear system of size $\mathtt{M}$ must be solved at each time step. When considering $\mathtt{P}$ time steps ($\mathtt{P}$ can reach several millions), the complexity grows very fast.
When considering POD-based model order reduction, the solution is projected into the reduced basis composed of functions $\{\phi_1(\mathbf{x}),\cdots,\phi_{\mathtt{R}}(\mathbf{x})\}$ extracted from some collected snapshots of the problem solution, with in general $\mathtt{R}\ll\mathtt{M}$, and consequently the solution approximation reads

$$u(\mathbf{x},t)\approx\sum_{i=1}^{\mathtt{R}}\xi_i(t)\phi_i(\mathbf{x}) \tag{2}$$

which requires the resolution of linear systems of size $\mathtt{R}$ instead of the ones of size $\mathtt{M}$ characteristic of finite element solutions. The use of a reduced basis implies in many cases impressive computing-time savings.
Approximations (1) or (2) imply a finite sum of time-dependent coefficients and space functions. The last are assumed known; they consist of the usual finite element shape functions or the modes extracted by applying, for example, Proper Orthogonal Decomposition – POD. A step forward could consist in assuming space functions to be also unknown and in computing both time and space functions on the fly. In this case, the approximation reads

$$u(\mathbf{x},t)\approx\sum_{i=1}^{\mathtt{N}}T_i(t)X_i(\mathbf{x}) \tag{3}$$

Because both functions involved in approximation (3) are unknown, it defines a nonlinear problem whose solution requires an appropriate linearization strategy. The interested reader can refer to [17] and the references therein for additional details on the separated representation constructor.
Expression (3) evidences that the solution procedure requires the resolution of about $\mathtt{N}$ problems, with $\mathtt{N}\ll\mathtt{M}$ and $\mathtt{N}\sim\mathtt{R}$ (in fact a bit more because of the nonlinearity induced by separated representations) involving the space coordinates (in general three-dimensional – 3D – and whose associated discrete systems are of size $\mathtt{M}$) for computing the space functions $X_i(\mathbf{x})$ and about $\mathtt{N}$ one-dimensional – 1D – problems for calculating the time functions $T_i(t)$. Due to the fact that the computing cost related to the solution of 1D problems is negligible with respect to the solution of 3D problems, the resulting computational complexity reduces drastically, scaling with $\mathtt{N}$ instead of $\mathtt{P}$.

(ii) *Space separation* allowed addressing multi-physics problems defined in degenerated geometries in which at least one of its dimensions remains much smaller that the other ones (e.g., beams, plates, shells, laminates...) or processes involving additive layers (e.g., automated tape placement, 3D printing, or additive manufacturing).

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