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An asymptotic Reissner–Mindlin plate model

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ABSTRACT

A mathematical study via variational convergence of a periodic distribution of classical linearly elastic thin plates softly abutted together shows that it is not necessary to use a different continuum model nor to make constitutive symmetry hypothesis as starting points to deduce the Reissner–Mindlin plate model.

1. Introduction

Due to its ability to account for shear effects, the Reissner–Mindlin plate model [1–3] is often preferred in the engineering literature (see [4]) over the Kirchhoff–Love plate model. So, as done for the Kirchhoff–Love plate model [5,6], it is challenging to proceed with a rigorous mathematical derivation of the Reissner–Mindlin plate model by studying the asymptotic behavior of a thin 3-dimensional elastic body when its thickness goes to zero. This was done in [7–9] by using a second gradient or Cosserat continuum for the body jointly with constitutive symmetry assumptions; here – being aware of the results of [10] on the bonding of thin plates – we prefer to consider a strongly heterogeneous *classical linearly elastic* body made of a periodic distribution of thin *anisotropic* plates abutted together.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{R}^3$ assimilated to the Euclidean physical space. For all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\hat{\xi}$ stands for (ξ_1, ξ_2). The space of all $(N \times N)$ symmetric matrices is denoted by $\mathbb{S}^N$ and equipped with the usual inner product and norm denoted by $\cdot$ and $|\cdot|$, as in $\mathbb{R}^3$. For all e in $\mathbb{S}^3$, we set

$$e = \hat{e} + e^{\perp} \tag{1}$$

where $(\hat{e})_{\alpha\beta} = e_{\alpha\beta}$ and $(e^{\perp})_{\alpha\beta} = 0$, $1 \le \alpha, \beta \le 2$, and $(\hat{e})_{i3} = 0$, $(e^{\perp})_{i3} = e_{i3}$, $1 \le i \le 3$. For all a, b in $\mathbb{R}^3$, $a \otimes_s b$ stands for the symmetrized tensor product of a by b. Moreover, for all subset $\mathcal{O}$ of $\mathbb{R}^N$, $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$. Finally, we will use the symbol $\mathfrak{h}_n$ to denote n-dimensional Hausdorff measure and the letter C to introduce various constants which may vary from line to line.

Let ω be a domain of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\omega$ and $2\eta_0$, ε_0 two positive real numbers lesser than 1. Let $Y := (0,1)^2$, $Y_\eta^{\text{ext}} := (-\eta, 1+\eta)^2$, $Y_\eta^{\text{int}} := (\eta, 1-\eta)^2$, $I_\varepsilon := \{i \in \mathbb{Z}^2 ;\ \varepsilon(i + Y_\eta^{\text{ext}}) \subset \omega\}$ for all η in $(0, \eta_0)$ and

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$$\omega^{\text{int}}_{\eta,\varepsilon} := \bigcup_{i\in I_\varepsilon} \varepsilon(i + Y^{\text{int}}_\eta), \quad \omega^{\text{ext}}_{\eta,\varepsilon} := \omega \setminus \bigcup_{i\in I_\varepsilon} (i + \overline{Y^{\text{ext}}_\eta})$$

$$\omega_{\eta,\varepsilon} := \omega^{\text{int}}_{\eta,\varepsilon} \cup \omega^{\text{ext}}_{\eta,\varepsilon}, \quad b_{\eta,\varepsilon} := \omega \setminus \overline{\omega_{\eta,\varepsilon}}$$

Let h be a small positive number, we will consider a structure occupying $\Omega^h := \omega \times (-h, h)$ made of an εY-periodic distribution of thin linearly elastic plates inhabiting $\Omega^{h\,\text{int}}_{\eta,\varepsilon} := \omega^{\text{int}}_{\eta,\varepsilon} \times (-h, h)$ abutted together through a thin and narrow soft linearly elastic adhesive layer filling $B^h_{\eta,\varepsilon} := b_{\eta,\varepsilon} \times (-h, h)$ and surrounded by a thin linearly hollow plate occupying $\Omega^{h\,\text{ext}}_{\eta,\varepsilon} := \omega^{\text{ext}}_{\eta,\varepsilon} \times (-h, h)$. We set $\Omega^h_{\eta,\varepsilon} := \Omega^{h\,\text{int}}_{\eta,\varepsilon} \cup \Omega^{h\,\text{ext}}_{\eta,\varepsilon}$ and assume that all the constituents of the structure are perfectly bonded together.

For brevity and simplicity, we assume that the structure is subjected to body and surface forces on its upper/lower boundary $\Gamma^h_\pm := \omega \times \{\pm h\}$ of densities f^h, g^h, respectively, and, as in [7,8], is clamped on its lateral boundary $\Gamma^h_{\text{tat}} := \partial\omega \times (-h, h)$. The strain energy $\mathcal{W}_s$ of the body reads as:

$$\mathcal{W}_s(x, e) := \begin{cases} \mathcal{W}(e) \text{ a.e. } x \in \Omega^h_{\eta,\varepsilon} \\ \mu_\wedge \mathcal{W}_\wedge(\hat{e}) + \mu_\perp \mathcal{W}_\perp(e^\perp) \text{ a.e. } x \in B^h_{\eta,\varepsilon} \end{cases}$$

for all e in $\mathbb{S}^3$, $\mathcal{W}$, $\mathcal{W}_\wedge$, $\mathcal{W}_\perp$ being positive definite quadratic functions on $\mathbb{S}^3$.

Hence the equilibrium of the structure involves a quintuplet of data $s := (s', \varepsilon)$, $s' := (\mu_\wedge, \mu_\perp, \eta, h)$ and reads as:

$$(\mathcal{P}^s) \quad \text{Min} \left\{ \int_{\Omega^h} \mathcal{W}_s(x, e(v))\,\mathrm{d}x - \int_{\Omega^h} f^h(x) \cdot v(x)\,\mathrm{d}x - \int_{\Gamma^h_+ \cup \Gamma^h_-} g^h(x) \cdot v(x)\,\mathrm{d}\mathfrak{H}_2 \,;\ v \in H^1_{\Gamma^h_{\text{tat}}}(\Omega^h; \mathbb{R}^3) \right\}$$

where $e(v)$ is the strain tensor associated with the displacement v and, in the sequel, for all domain $\mathcal{O}$ in $\mathbb{R}^N$ and all smooth part γ of its boundary $\partial\mathcal{O}$, $H^1_\gamma(\mathcal{O}; \mathbb{R}^N)$ denotes the subspace of $H^1(\mathcal{O}; \mathbb{R}^N)$ made of the elements with vanishing trace on γ. Clearly, if (f^h, g^h) belongs to $L^2(\Omega^h \times (\Gamma^h_+ \cup \Gamma^h_-); \mathbb{R}^3)$, $(\mathcal{P}^s)$ has a unique solution u^s and, considering the data s as a parameter, we are interested in its asymptotic behavior when s takes values in a countable subset of $(0, +\infty)^2 \times (0, \eta_0) \times (0, +\infty) \times (0, \varepsilon_0)$ with 0 as a unique limit point. Like in the mathematical derivation of the Kirchhoff–Love theory of plates [5,6], it is convenient to introduce the linear mappings Π^h and S_h:

$$\xi = (\hat{\xi}, \xi_3) \in \mathbb{R}^3 \mapsto \Pi^h \xi = (\hat{\xi}, h\xi_3)$$

$$v \in L^1(\Omega^h; \mathbb{R}^3) \mapsto S_h v \in L^1(\Omega; \mathbb{R}^3) \text{ s.t. } (S_h v)(x) = \frac{1}{h} \Pi^h(v(\Pi^h x)),\ \forall x \in \Omega := \omega \times (-1, 1)$$

We make the following assumption on the loading:

$$(H_1) \quad : \quad \begin{cases} \exists (f, f', g) \in L^3(\Omega; \mathbb{R}^3) \times L^3(\Omega; \mathbb{R}^3) \times L^\infty(\Gamma^+ \cup \Gamma^-; \mathbb{R}^3) \text{ s.t.} \\ f^h(\Pi^h x) = h\Big(\chi_{\Omega^h_{\eta,\varepsilon}} \Pi^h f(x) + \chi_{B^h_{\eta,\varepsilon}} \Pi^h f'(x)\Big),\ \forall x \in \Omega \\ g^h(\Pi^h x) = h^2 \Pi^h\, \chi_{\Gamma_{\eta,\varepsilon,+} \cup \Gamma_{\eta,\varepsilon,-}} g(x),\ \forall x \in \Gamma_{\eta,\varepsilon,+} \cup \Gamma_{\eta,\varepsilon,-} \end{cases}$$

where $\Gamma_{\eta,\varepsilon,\pm} := \omega_{\eta,\varepsilon} \times \{\pm 1\}$. Therefore, $u_s := S_h u^s$ is the unique solution to

$$(\mathcal{P}_s) \quad \text{Min} \left\{ \mathcal{J}_s(v);\ v \in H^1_{\Gamma_{\text{tat}}}(\Omega; \mathbb{R}^3) \right\}$$

where

$$\mathcal{J}_s(v) := \int_\Omega \mathcal{W}_s(x, e(h, v)) - (\chi_{\Omega_{\eta,\varepsilon}} f + \chi_{B_{\eta,\varepsilon}} f') \cdot v\,\mathrm{d}x - \int_{\Gamma_{\eta,\varepsilon,+} \cup \Gamma_{\eta,\varepsilon,-}} g \cdot v\,\mathrm{d}\mathfrak{H}_2$$

$$e(h, v)_{\alpha\beta} := e(v)_{\alpha\beta}, \quad e(h, v)_{\alpha 3} := \frac{1}{h} e(v)_{\alpha 3},\ 1 \le \alpha, \beta \le 2, \quad e(h, v)_{33} := \frac{1}{h^2} e(v)_{33}$$

with Γ_{tat} the reciprocal image by Π^h of Γ^h_{tat} and, similarly, index h will be dropped for the image by $\Pi^{h^{-1}}$ of $\Omega^h_{\eta,\varepsilon}$, $B^h_{\eta,\varepsilon}$ and $\Gamma^h_\pm$.