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Disk in a groove with friction: An analysis of static equilibrium and indeterminacy

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ABSTRACT

This note studies the statics of a rigid disk placed in a V-shaped groove with frictional walls and subjected to gravity and a torque. The two-dimensional equilibrium problem is formulated in terms of the angles that contact forces form with the normal to the walls. This approach leads to a single trigonometric equation in two variables whose domain is determined by Coulomb's law of friction. The properties of solutions (existence, uniqueness, or indeterminacy) as functions of groove angle, friction coefficient and applied torque are derived by a simple geometric representation. The results modify some of the conclusions by other authors on the same problem.

1. Introduction

The problem of the equilibrium of a rigid disk wedged in a V-shaped groove with frictional walls is of interest in the study of granular packing, as it represents an elementary model of particle contact. In [1], the disk was considered to be held by gravity in an inclined groove; in other studies [2,3], the groove was vertical, but the disk was subjected to more general forces. Both configurations lead to an indeterminate problem, because four unknowns (the components of contact forces) are involved, but only three equations are available.

For the frictional vertical groove, McNamara et al. [2] gave a detailed analytical discussion of the disk equilibrium under the action of gravity and a torque. The presence of indeterminacy was related to the applied forces, the groove angle, and the friction coefficient; some of the results of [2] were also summarized by Stamm [4], pp. 12–14.

The aim of this note is to propose a different and possibly simpler treatment of the configuration discussed by McNamara et al. [2], by using the xy-components and angles of contact forces rather than their normal/tangential projections. This choice leads to a trigonometric equation in two variables, whose solutions are discussed with the aid of a geometrical scheme.

2. Equilibrium equations

A homogeneous rigid disk of radius r, center O and weight $m\mathbf{g}$ lies in a groove of aperture angle 2θ; the contacts A and B between the disk and the groove walls have friction, with static coefficient $\mu < 1$. In addition to gravity, the disk is acted upon by a torque $M > 0$ (torque and angles are taken positive in the counterclockwise direction). Disk equilibrium

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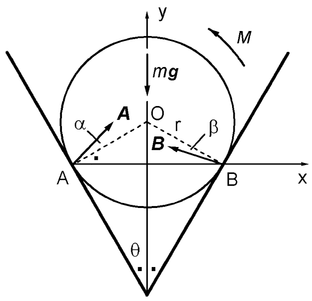

Fig. 1. Scheme of disk in a groove with frictional walls.

requires that the resultant force and total moment, for instance about O, be zero. In the xy reference system of Fig. 1, the components of the contact forces **A** and **B** must then obey the linear equations

$$A_x + B_x = 0 \tag{1a}$$

$$A_y + B_y - mg = 0 \tag{1b}$$

$$(-A_y + B_y)r\cos\theta + M = 0 \tag{1c}$$

Equations (1b) and (1c) determine the values of A_y and B_y; by introducing the non-dimensional, normalized torque

$$\tau = \frac{M}{mgr\cos\theta} \tag{2}$$

which is positive, since in a groove $0 < \theta < \pi/2$, the solution for A_y and B_y can be written in the form

$$A_y = \frac{1}{2}mg(1+\tau) \tag{3a}$$

$$B_y = \frac{1}{2}mg(1-\tau) \tag{3b}$$

The components A_x and B_x appear only in Equation (1a), which yields $B_x = -A_x$, but the value of A_x cannot be established, so that the problem is in general indeterminate. By converting the components of the four-dimensional contact forces used in [2] into their xy equivalents, it can be proved that the undetermined coefficient a_0 introduced there is the same as $2A_x$.

Although the contact forces **A** and **B** cannot be uniquely specified, they must still comply with Coulomb's law of friction. This law sets a maximum magnitude γ to the angles of inclination α and β of these forces to the respective wall normal. The value $\mu = \tan\gamma$ is the static friction coefficient: here the assumption $\mu < 1$ entails $\gamma < \pi/4$. As suggested by Fig. 1, here both α and β must be positive or zero. In fact, for negative α (or β) the frictional force **A** (or **B**) would produce a moment about O with the same sense as M, in contradiction with friction's oppositional nature (see, e.g., rule #1 in Goodman and Warner [5], page 286). Hence Coulomb's law yields the inequalities:

$$0 \le \tan\alpha \le \mu, \qquad 0 \le \tan\beta \le \mu \tag{4}$$

We now note that B_x is certainly ≤ 0, since $0 \le \beta \le \gamma$ (Fig. 1); from equation (1a), we have $A_x = -B_x = |B_x|$, hence $A_x \ge 0$. The relation between α, β and the components of **A**, **B** then takes the form (see Fig. 1)

$$\tan(\theta+\alpha) = A_y/A_x \tag{5a}$$

$$\tan(\theta-\beta) = B_y/|B_x| \tag{5b}$$

under the assumption that $A_x \ne 0$ (the case $A_x = 0$ is considered in section 4.1). By taking the ratio of equation (5b) to (5a) and recalling that $|B_x| = A_x$, we can eliminate the unknown component A_x. Then we replace in this ratio the respective expressions for A_y and B_y given in (3a) and (3b), and finally get

$$\frac{\tan(\theta-\beta)}{\tan(\theta+\alpha)} = k \tag{6}$$

where

$$k = \frac{1-\tau}{1+\tau} \tag{7}$$

In equation (6) the contact forces at A and B appear through the angles α and β, and the forces imposed on the disk are condensed into the parameter k of (7), with τ given by (2). The absolute value of k is always less than 1; k is near unity