Blow-up of solutions to a quasilinear wave equation for high initial energy

Fang Li [a], Fang Liu [b],*

[a] *School of Mathematics, Jilin University, Changchun 130012, PR China*
[b] *School of Basic Sciences, Changchun University of Technology, Changchun 130012, PR China*

A B S T R A C T

This paper deals with blow-up solutions to a nonlinear hyperbolic equation with variable exponent of nonlinearities. By constructing a new control function and using energy inequalities, the authors obtain the lower bound estimate of the L^2 norm of the solution. Furthermore, the concavity arguments are used to prove the nonexistence of solutions; at the same time, an estimate of the upper bound of blow-up time is also obtained. This result extends and improves those of [1,2].

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1. Introduction

In this paper, we consider the following quasilinear hyperbolic problem

$$\begin{cases} u_{tt} - \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) - \Delta u_t = |u|^{q(x)-2}u, & (x,t) \in \Omega \times (0,T) := Q_T \\ u(x,t) = 0, & (x,t) = \partial\Omega \times (0,T) := \Gamma_T \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1.1}$$

where $\Omega \subset R^N (N \geqslant 1)$ is a bounded domain, $\partial\Omega$ is Lipschitz continuous. It will be assumed throughout the paper that the exponents $p(x), q(x)$ satisfy the following conditions.

$$2 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty, \ 1 < q^- \leqslant q(x) \leqslant q^+ < \infty$$

Problem (1.1) models many physical problems such as viscoelastic fluids, electro-rheological fluids, processes of filtration through a porous media, fluids with temperature-dependent viscosity, etc. The interested readers may refer to [3–5] and the references therein. When p, q are fixed constants, many authors discussed the existence of solutions, finite-time blow-up of solutions for low initial energy and arbitrarily high initial energy as well as some estimates of a lower bound for blow-up times – the interested readers may refer to [6–14]. When p, q are continuous functions, S.N. Antontsev in [1,15] discussed the blowing-up properties of solutions to the initial and homogeneous boundary value problem of quasilinear wave equations involving the $p(x,t)$-Laplacian operator and a negative initial energy. Guo–Gao of [2] proved that the solution blew up in finite time for positive initial energy. Later, Guo, in [16], applied the interpolation inequality and energy inequalities to obtain an estimate of the lower bound for the blow-up time when the source is super-linear. In addition, Messaoudi and

* Corresponding author.
  E-mail addresses: lf8583@jlu.edu.cn (F. Li), fairfang@sina.com (F. Liu).

Talahmeh in [17,18] discussed blow-up properties of solutions to the nonlinear wave equations with weak damping terms and a $p(.)$-Laplacian operator. However, there are few works that discuss blow-up properties of solutions for high initial energy. In fact, dealing with such problems, one has to face some difficulties:

• how can one give an estimate of the lower bound of the norm $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$? In fact, due to the presence of strong damping term Δu_t, the technique used in [17] is not applicable, so, we have to find some new methods or techniques;

• how can one establish the quantitative relationship between the term $\int_\Omega |u|^{q(\cdot)}dx$ and the initial energy? Since the initial energy is arbitrary, we can not apply the technique used in [2] to give the quantitative relation between $\int_\Omega |u|^{q(\cdot)}dx$ and $E(0)$.

In this paper, we construct a new control function and apply energy estimate inequalities to bypass the first difficulty above. Furthermore, by modifying the functional constructed in [2] and utilizing the quantitative relationship between the term $\int_\Omega |u|^2 dx$ and the initial energy, we prove that the solution blows up in finite time for arbitrary positive initial energy. In particular, it is worth pointing out that our results extend and improve those of [1,2]. For the existence of solutions, we may refer to [1,2,15].

Define the energy functional as the following:

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 dx + \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx$$

First of all, due to $p_t = q_t = 0$ and $a(x,t) = b(x,t) = 1$, we follow the line of the proof of Lemma 2.1 in [1] or Lemma 1.1 of [2] to obtain the energy functional $E(t)$, which satisfies the following identity.

Lemma 1.1. *Suppose that* $u \in L^{q(x)}(Q_T) \cap L^\infty(0,T; W_0^{1,p(x)}(\Omega))$, $u_t \in L^2(0,T; H^1(\Omega))$ *is a solution to Problem* (1.1), *then* $E(t)$ *satisfies the following identity:*

$$E(t) + \int_0^t \int_\Omega |\nabla u_s|^2 dx\, ds = E(0), \quad t \geqslant 0 \tag{1.2}$$

if the following conditions are fulfilled

$$\begin{cases} 2 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty, \ 1 < q^- \leqslant q(x) \leqslant q^+ < p^{-*} \\ p^{-*} = \begin{cases} \frac{Np^-}{N-p^-}, & \text{if } 1 < p^- < N \\ \infty, & \text{if } p^- \geqslant N \end{cases} \\ u_0 \in W_0^{1,p(x)}(\Omega) \cap L^{q(x)}(\Omega), \ u_1 \in L^2(\Omega) \end{cases} \tag{1.3}$$

Let λ_1 be the first eigenvalue of the following problem

$$\begin{cases} -\Delta\psi = \lambda\psi, & x \in \Omega \\ \psi = 0, & x \in \partial\Omega \end{cases}$$

and set $B_1 = \min\{\lambda_1, \frac{p^+(q^-+2)}{2(q^--p^+)}\}$.

Lemma 1.2. *If all the conditions of Lemma* 1.1 *remain true and* $q^- > p^+$, *then the solution to Problem* (1.1) *satisfies the following inequality*

$$\int_\Omega uu_t dx \geqslant e^{M_0 t}\left[\int_\Omega u_0 u_1 dx - \frac{q^-}{M_0}E(0) - \frac{|\Omega|}{M_0}\right] + \frac{q^-}{M_0}E(t) + \frac{|\Omega|}{M_0}, \quad t > 0 \tag{1.4}$$

where $M_0 = \frac{4(q^--p^+)q^-B_1}{p^+(B_1+4q^-)} > 0$.