ARTICLE IN PRESS
CRAS2B:3566

C. R. Mecanique ••• (••••) •••–•••

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The stochastic energy-Casimir method

La méthode d'énergie-Casimir stochastique

Alexis Arnaudon, Nader Ganaba *, Darryl D. Holm

Department of Mathematics, Imperial College, London SW7 2AZ, UK

ARTICLE INFO

Article history:
Received 9 November 2017
Accepted 18 January 2018
Available online xxxx

Keywords:
Stochastic geometric mechanics
Energy-Casimir method
Stochastic stability

Mots-clés :
Mécanique géométrique stochastique
Méthode d'énergie-Casimir
Stabilité stochastique

ABSTRACT

In this paper, we extend the energy-Casimir stability method for deterministic Lie–Poisson Hamiltonian systems to provide sufficient conditions for stability in probability of stochastic dynamical systems with symmetries. We illustrate this theory with classical examples of coadjoint motion, including the rigid body, the heavy top, and the compressible Euler equation in two dimensions. The main result is that stable deterministic equilibria remain stable in probability up to a certain stopping time that depends on the amplitude of the noise for finite-dimensional systems and on the amplitude of the spatial derivative of the noise for infinite-dimensional systems.
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RÉSUMÉ

Dans cet article, nous étendons la méthode d'énergie-Casimir de stabilité des systèmes déterministes hamiltoniens de Lie–Poisson afin de fournir des conditions suffisantes de stabilité en probabilité des systèmes dynamiques stochastiques par des symétries. Nous illustrons cette théorie par des exemples classiques de mouvements coadjoints, comme le corps solide, la toupie pesante et l'équation d'Euler compressible en deux dimensions. Le principal résultat de cette extension est que les équilibres relatifs déterministes stables restent stables en probabilité jusqu'à un certain temps d'arrêt. Ce dernier dépend, d'une part, de l'amplitude du bruit pour les systèmes de dimensions finies et, d'autre part, de l'amplitude de la dérivée spatiale du bruit pour les systèmes de dimensions infinies.
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1. Introduction

In 1966, V. I. Arnold's fundamental paper [1] showed that ideal fluid mechanics can be cast into a geometric framework. In this framework of differential geometry and Lie group symmetry, the mathematical properties of ideal (nondissipative)

* Corresponding author.
E-mail addresses: alexis.arnaudon@imperial.ac.uk (A. Arnaudon), nader.ganaba13@imperial.ac.uk (N. Ganaba), d.holm@imperial.ac.uk (D.D. Holm).

https://doi.org/10.1016/j.crme.2018.01.003
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Please cite this article in press as: A. Arnaudon et al., The stochastic energy-Casimir method, C. R. Mecanique (2018), https://doi.org/10.1016/j.crme.2018.01.003

classical fluid mechanical systems are easily identified. For example, Arnold's geometric interpretation [1] of ideal incompressible fluid dynamics as geodesic motion on the group of diffeomorphisms was soon followed by a series of fundamental results in analysis, e.g., in [2]. We shall be interested here in another development of Arnold's geometric approach to fluid dynamics, which concerns the nonlinear stability of equilibrium (time-independent) solutions. A nonlinear fluid stability method based on the Lyapunov method was already introduced in the early days of geometric mechanics by Arnold in [3,4] for ideal incompressible fluid flows, whose L^2 kinetic energy norm provides the metric for their geodesic interpretation. This approach was extended in [5] to the energy-Casimir method, which allows for both kinetic and potential energy contributions and, hence, may be applied to a large class of ideal mechanical systems. This class of systems comprises Hamiltonian systems that admit reduction by Lie group symmetries. Such systems possess Lie–Poisson brackets whose null eigenvectors correspond to variational derivatives of conserved quantities called Casimirs. The Casimirs commute under the Lie–Poisson bracket with any functionals on the symmetry-reduced space, as well as with the system Hamiltonian itself. For Hamiltonian systems that do not have a Casimir function, the energy-momentum method, developed in [6], is used instead. This method uses momentum maps instead of the Casimirs to obtain stability results.

Interest has been growing recently in stochastic perturbations of mechanical systems with symmetries whose dynamics can be investigated in the framework of geometric mechanics. The aim of this new science of stochastic geometric mechanics is to extend to stochastic systems the mathematical understanding gained for deterministic systems by using differential geometry and Lie groups. The theory of stochastic canonical Hamiltonian systems began with Bismut [7], and was recently updated in geometric terms in [8]. This theory was extended to stochastic ideal fluid dynamics in [9] by using a Lie-group symmetry reduction of a stochastic Hamilton principle. The general theory was developed and illustrated further for finite-dimensional Euler–Poincaré variational principles with symmetry, leading to noncanonical stochastic Hamiltonian mechanical systems in [10,11].

The present work will seek sufficient conditions for the probabilistic stability of critical points of stochastic geometric mechanics systems, by using an extension of the energy-Casimir method. For this endeavour, we will need to introduce an appropriate notion of stability in probability; so that a stochastic counterpart of the energy-Casimir method can be developed and applied to stochastic dynamical systems. The main result of this paper is the proof that a deterministically stable stationary solution remains stable in probability up to a finite stopping time, for multiplicative stochastic perturbations that preserve coadjoint orbits. This theorem applies only if unique solutions to the stochastic process exist. However, since the stability in probability is valid only for finite time, existence and uniqueness of solutions is only needed locally in time.

Plan of the paper. Section 2 reviews the theory of stochastic perturbations of mechanical systems with symmetries developed by [9,10]. It also distinguishes between the notions of stability in the deterministic and stochastic settings, in the context of the deterministic energy-Casimir method. Section 3 forms the core of the paper, in which the stochastic energy-Casimir method is developed. Section 4 then illustrates the stochastic modifications of the energy-Casimir stability analysis for several classical examples, including the rigid body, the heavy top, and compressible Euler equations.

2. Preliminaries

2.1. Stochastic mechanical systems with symmetries

This section begins with defining the type of stochastic perturbations of mechanical systems that we will study in this work. For further detail, we refer the interested reader to [10,11] for finite-dimensional systems and to [9] for infinite-dimensional systems. Although the theory has been studied quite generally in [11], here we will restrict ourselves to the examples in [10] and [9]. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. For a probability space $(\Omega, \mathcal{F}_t, P)$, we consider a Wiener process W_t defined with respect to the standard filtration $\mathcal{F}_t$. The construction is based on the following stochastic Hamilton–Pontryagin variational principle: $\delta S = 0$ for the stochastic action integral,

$$S(\xi, g, \mu) = \int l(\xi)\,\mathrm{d}t + \int \left\langle \mu, \circ g^{-1}\mathrm{d}g - \xi\,\mathrm{d}t + \sum_i \sigma_i \circ \mathrm{d}W_t^i \right\rangle \tag{1}$$

In this formula, $g \in G$, $\xi \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ under the non-degenerate pairing $\langle \cdot, \cdot \rangle$. The vector fields $\sigma_i \in \mathfrak{g}$ represent constant multiples of Lie algebra basis elements and the symbol $\circ$ denotes Stratonovich stochastic integrals. The action integral (1) is invariant under left translations of the group G. We refer the reader to [12–15] for more details on the Hamilton–Pontryagin principle and to [16,15] for the use of Lie groups. Upon taking free variations $\delta\xi, \delta\mu$ and δg, and rearranging the terms, we find the momentum map relation $\frac{\delta l}{\delta \xi} = \mu$ and the Euler–Poincaré equation for its stochastic coadjoint motion,

$$\mathrm{d}\mu = \mathrm{ad}^*_{(\circ g^{-1}\mathrm{d}g)}\mu = \mathrm{ad}^*_{\xi}\mu\,\mathrm{d}t + \mathrm{ad}^*_{\sigma_i}\mu \circ \mathrm{d}W_t^i \tag{2}$$

where the relation $\circ g^{-1}\mathrm{d}g = \xi\,\mathrm{d}t - \sum_i \sigma_i \circ \mathrm{d}W_t^i$ for the left-invariant reduced velocity is imposed by variations with respect to μ, regarded as a Lagrange multiplier. Thus, the solutions to the stochastic Euler–Poincaré equation (2) preserve coadjoint orbits, even in the presence of noise. Besides the coadjoint orbits, other quantities conserved by the stochastic

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