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The legacy of Jean-Jacques Moreau in mechanics

Moreau's hydrodynamic helicity and the life of vortex knots and links

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ABSTRACT

This contribution to an issue of *Comptes rendus Mécanique*, commemorating the scientific work of Jean-Jacques Moreau (1923–2014), is intended to give a brief overview of recent developments in the study of helicity dynamics in real fluids and an outlook on the growing legacy of Moreau's work. Moreau's discovery of the conservation of hydrodynamic helicity, presented in an article in the *Comptes rendus de l'Académie des sciences* in 1961, was not recognized until long after it was published. This seminal contribution is gaining a new life now that modern developments allow the study of helicity and topology in fields and is having a growing impact on diverse areas of physics.

Conservation laws, such as conservation of energy, underpin our most fundamental understanding of physical phenomena. The equations of fluid motion are a prime example: themselves an expression of conservation of momentum, mass and energy, they capture the myriad of fluid motions we see in the world. It is thus a rare privilege indeed to discover an additional conservation law. Jean-Jacques Moreau achieved just this in his seminal work of 1961 [1], by showing that within the Euler equations of fluid motion there lurked an additional conservation law: that of hydrodynamic helicity H:

$$H = \int u \cdot \omega \, \mathrm{d}V$$

where u is the flow velocity and $\omega = \nabla \times u$ the vorticity. Moreau's work followed a similar discovery in plasma physics by Woljer [2], and was followed by another seminal contribution by Keith Moffatt [3], who established the topological interpretation of both results (see Moffatt's contribution to this issue for a more precise account of events).

It is surprising that a new conservation law could be discovered so long after the Euler equations of motion. The origins of this long interval lie perhaps in the subtle geometric nature of helicity. Unlike momentum and energy, helicity does not originate in common space–time symmetries, such as translational symmetry or time-translation symmetry. Rather, as shown by Moffatt [3], helicity is intimately related to the topology of the flow.

This connection between helicity and topology can be intuitively understood by the following simple example: consider a thin vortex ring in which all the vorticity is parallel to the tube axis and lies within a tube D, having centerline c and cross section A, as depicted in Fig. 1. In this case the helicity simplifies to:

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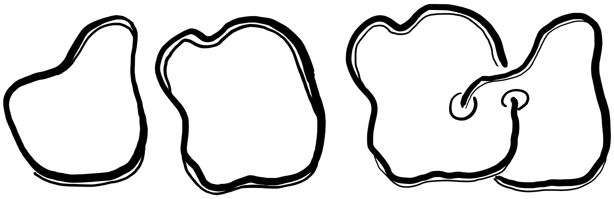

Fig. 1. Linked vortex lines carry helicity whereas unlinked vortex lines have zero helicity.

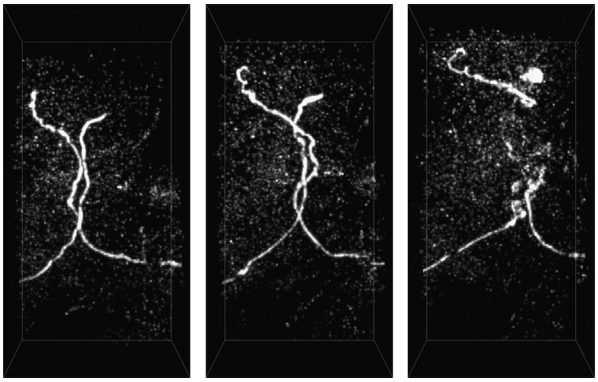

Fig. 2. A reconnection between vortex strands in water (from Kleckner and Irvine [8]). The image shows light scattered off air bubbles trapped in the core of the reconnecting vortex strands; the strands approach each other and re-connect.

$$H = \int_D u \cdot \omega \, \mathrm{d}V = \Gamma \oint_c u \cdot \mathrm{d}l$$

where $\Gamma = \int_A \omega \, \mathrm{d}A$. By Stokes' theorem, the right-hand side will be equal to the amount of flux that threads the ring, which is zero for disconnected rings. If, however the rings are linked, then:

$$H = 2\,\Gamma^2$$

This intuitive idea generalizes to more intricate vortex geometries as well as continuous fields. The value $2\,\Gamma^2$ is in fact $2L_{1,2}\Gamma_1\Gamma_2$, where $L_{1,2}$ is the Gauss linking number of the two vortices, and it is multiplied by their respective circulations. For a general field, seen as a collection of N infinitesimal flux tubes, we have [4,5]:

$$H = \sum_{i,j}^{N\to\infty} \Gamma_i \Gamma_j L_{ij}$$

Helicity is thus the circulation-weighted sum of the topological linking number between all vortex line pairs.

Taking this elegant topological interpretation as a starting point, the conservation of helicity then follows simply from Helmholtz's laws of vortex motion that vorticity goes with the flow [6]: since vortex lines are transported by the velocity field, they cannot cross. This notion of topological robustness is exactly the concept that inspired Lord Kelvin's hypothesis that atoms are knotted vortices in the aether. It wasn't in fact till much later that this type of topological invariant was rationalized as a particle-relabeling symmetry and connected to the more commonly encountered circulation theorems [7].

This simple view of the conservation of helicity, is, however, shattered in the presence of viscosity, which mediates reconnection processes in which vortex lines are cut and spliced together in reconnections as shown in Fig. 2. Investigating the validity of Moreau's and Moffatt's law in real physical systems is difficult theoretically, as well as numerically, because of the singular nature of reconnections and the large scale separation often involved. Experimentally preparing vortex knots – the quintessential helicity-bearing excitations – had also remained an outstanding challenge since Lord Kelvin's conjecture, until recent advances.

In the past decade or so, a number of tools have changed the landscape. On the experimental side, additive manufacturing (3D printing) has opened the way of rapidly and cheaply creating objects with arbitrary shapes. Three-dimensional imaging and data-processing has become almost routine. Furthermore, many results of differential geometry and topology have percolated to the basic education of physicists. These advances made it possible to create the first vortex knots and links shown in Fig. 3. A physical embodiment of Kelvin's, Moreau's, and Moffatt's vision.

Once made, such vortex knots immediately distort and disconnect through reconnections (Fig. 4), as if drawn towards topological simplicity. For example a vortex link immediately decays in to two separate rings. This penchant for unraveling

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