ARTICLE IN PRESS CRAS2B:3550

C. R. Mecanique ••• (••••) •••–•••

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The legacy of Jean-Jacques Moreau in mechanics

Recovering convexity in non-associated plasticity

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ARTICLE INFO

Article history:
Received 20 May 2017
Accepted 11 October 2017
Available online xxxx

Keywords:
Elasto-plasticity
Non-associated flow rules
Quasi-static evolution
Convex analysis

ABSTRACT

We quickly review two main non-associated plasticity models, the Armstrong–Frederick model of nonlinear kinematic hardening and the Drucker–Prager cap model. Non-associativity is commonly thought to preclude any kind of variational formulation, be it in a Hencky-type (static) setting, or when considering a quasi-static evolution because non-associativity destroys convexity. We demonstrate that such an opinion is misguided: associativity (and convexity) can be restored at the expense of the introduction of state variable-dependent dissipation potentials.

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1. Introductory remarks

Although small-strain Prandtl–Von Mises elastoplasticity is hardly a practical, or realistic model for elastoplastic evolutions, it is undoubtedly the theoretical template that any macroscopic departure from elasticity must elaborate upon when some kind of slip-friction mechanism is involved, be it at the mesoscopic scale as is the case for granular materials, or at the microscopic scale as in metals.

For a homogeneous elastoplastic material occupying a volume $\Omega \subset \mathbb{R}^3$, with Hooke's law (elasticity tensor) A subject to a time-dependent loading process with, say, $f(t)$ as body loads, $g(t)$ as surface loads on a part Γ_s of $\partial\Omega$, and $w(t)$ as displacement loads (hard device) on the complementary part Γ_d of $\partial\Omega$, the quasi-static model reads as detailed below.

- Kinematic compatibility: $Eu(t) := 1/2(\nabla u(t) + \nabla u^{\mathrm{T}}(t) = e(t) + p(t)$ in Ω and $u(t) = w(t)$ on Γ_d. Here $u(t)$ is the displacement field at t and $e(t)$ and $p(t)$ (a deviatoric symmetric matrix, that is an element of the space $\mathbb{M}^{3\times 3}_D$ of 3×3-trace free symmetric matrices) are the elastic and plastic strains at t.
- Equilibrium: $\mathrm{div}\sigma(t) + f(t) = 0$ in Ω and $\sigma(t)\nu = g(t)$ on Γ_s, where ν denotes the outer unit normal to $\partial\Omega$. Here $\sigma(t)$ is the Cauchy stress tensor at time t.
- Hooke's law: $\sigma(t) = Ae(t)$ in Ω.
- Stress constraint: $\sigma_D(t) \in K$[1] and K is the admissible set of stresses (a convex and compact subset of symmetric deviatoric $n \times n$ matrices that contains 0 in its interior).
- Normality rule: $\dot{p}(t) = 0$ if $\sigma_D(t) \in \mathrm{int}\, K$, while $\dot{p}(t)$ belongs to the normal cone to K at $\sigma_D(t)$ if $\sigma_D(t) \in \partial K$.

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[1] Here and in all that follows the subscript $_D$ affixed to a symmetric matrix denotes the projection of that matrix onto the subspace of trace free matrices, while the dot above any time-dependent quantity refers to the time derivative of that quantity.

https://doi.org/10.1016/j.crme.2017.12.005
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Please cite this article in press as: G.A. Francfort, Recovering convexity in non-associated plasticity, C. R. Mecanique (2017), https://doi.org/10.1016/j.crme.2017.12.005

The set of equations described above is actually attuned to the behavior of metals, or more generally, of crystalline solids, because the deviatoric character of the plastic strain is then viewed as the macroscopic manifestation of dislocations. When dealing with materials such as rock, sand or concrete, it is more appropriate to view the set of admissible stresses K as just a convex, possibly unbounded set of symmetric $n \times n$ matrices, because plastic dilations are actually evidenced in such materials.

In any case, this model has been a seemingly inexhaustible source of mechanical as well as mathematical exegeses, starting with the pioneering work of Jean-Jacques Moreau, who recognized and formalized the fundamental role that convexity and duality should play in any rational handling of elastoplasticity.

Unfortunately, in plasticity as elsewhere, complexity soon prevails.

When it comes to metals, it first rears its head in the form of hardening, the realization that the convex K is not a fixed set, but rather that it evolves with the history of the loading process. For metals, it usually grows "larger", making the metal "harder". From a mathematical standpoint, the resulting complexification is paradoxically beneficial because, be it for isotropic or kinematic hardening (for which the convex K actually translates with the plastic strain), the functional analytic environment of the new model is more classical. There is no need to resort to spaces where the symmetrized gradient is a measure and the evolution runs in classical Sobolev spaces; see, e.g., [1, Section 4] in the case of linear isotropic hardening.

The introduction of hardening is soon seen to be insufficient to fully explain well-documented effects like the Bauschinger effect, which shows a reduction in admissible compressive strength after a material is pulled in tension. Indeed linear kinematic hardening fails to produce the observed smooth transition from elastic unloading to plastic hardening in the compressive part of the hysteresis diagram [2, Figure 2(a)]. To account for such a behavior, P.J. Armstrong and C.O. Frederick [3] proposed a kinematic hardening model for which the normality rule does not hold. That model, later modified and popularized most notably by J.-L. Chaboche (see, e.g., [4]), has become the fulcrum of a slew of evermore sophisticated models that attempt to account for the many quirks of metal behavior under cyclic loading.

When it comes to dirtier materials such as soils or concrete, it shows up in the misfit between the angle of friction φ, which determines the shape of the set of admissible stresses – a cone with the hydrostatic stress as the axis, a positive hydrostatic stress as the summit (corresponding to maximal three-axial traction) and aperture $\arctan(\sin\varphi)$ – and the angle of dilatancy ψ, which corresponds to the aperture of a cone of the same type with the direction of the plastic flow[2] as the normal. It is usually such that $\varphi > \psi$ [5], and then the normality rule no longer holds. Such is the case for the Drucker–Prager-, or Mohr–Coulomb-type models that are widely used in soil or rock mechanics [6,7].

As noted, the striking feature shared by the Armstrong–Frederick-type models and the Drucker–Prager- or Mohr–Coulomb-type models is the failure of normality. In the terminology of elastoplasticity, they are *non-associated* models. The associated mathematical pathology is a lack of convexity. Indeed, classically, that is in the case of perfect plasticity, the enforcement of the Clausius–Duhem inequality – itself a rewriting of the second law of thermodynamics – imposes that

$$\sigma_{(D)}(t) \cdot \dot{p}(t) \geq 0 \tag{1.1}$$

throughout the evolution. This is precisely what normality achieves, since the normality rule equivalently reads as $\dot{p}(t) \in \partial I_K(\sigma_{(D)}(t))$, where I_K is the indicatrix function of the set K, so that, by elementary convex analysis and since $0 \in \operatorname{int} K$, (1.1) holds.[3] Alternatively, one may view normality as equivalent to Drucker–Ilyushin's postulate [8,9].[4] In the absence of normality, inequality (1.1) comes as a restriction on the possible K's and flow rules. Note that, since $\varphi > \psi$ Clausius–Duhem is automatically satisfied in the Drucker–Prager-, or Mohr–Coulomb-type models, at least when $\psi \geq 0$.[5]

In any case, popular wisdom has it that non-associated flow rules result in a system of equations that cannot fit within any kind of variational framework. Such a vague statement cannot be accepted at face value. It could be interpreted as the inability to derive the analogue of a Hencky plasticity-type variational formulation. In a more modern setting, it could also be seen as the inability to formulate a variational evolution similar to that in [10]. We will not go down that route and refer the interested reader to [10] for an exposition of the variational evolution in perfect elastoplasticity and to [11–13] for its extension to the non-associative framework.

Rather, we will simply demonstrate here that non-associated flow rules can be rendered associative at the expense of the addition of a state variable in the dissipation potential. Once this is done, it should be intuitively clear that both Hencky and evolutionary plasticity can be framed variationally. Of course, the mathematical hurdles might prove overwhelming, but this is not the story that we wish to tell here.

The first section will be devoted to the Armstrong–Frederick model introduced in [3], while the second section will be focussing on the Drucker–Prager and Mohr–Coulomb models [7]. The corresponding detailed works are [13] and [12]. Other non-associated models could be treated similarly. In particular, the reader is referred to [14,11,15] for the case of Cam–Clay plasticity.

[2] $\sin\psi$ is also the ratio of the volumetric to axial strain rates in a triaxial test [5].

[3] Notationwise, if $x \in \mathbb{R}^N \mapsto g(x) \in \mathbb{R} \cup \infty$ is a convex function, we denote throughout by $\partial g(x)$ the sub-differential set of g at x.

[4] As an aside, while normality is equivalent to Ilyushin's principle, it is strictly more restrictive than Clausius–Duhem's inequality, as demonstrated in [9].

[5] The case $\psi < 0$, while worth investigating if thinking of, e.g., loose sands, is not within the purview of the present study, because it leads to softening. Softening, which forces a drastic departure from convexity, lies beyond the reach of the method expounded in this work.

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