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The legacy of Jean-Jacques Moreau in mechanics

An overview of the formulation, existence and uniqueness issues for the initial value problem raised by the dynamics of discrete systems with unilateral contact and dry friction

Une revue des questions de formulation, existence, unicité, soulevées par la dynamique des systèmes discrets en présence de contact et frottement sec

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A B S T R A C T

In the end of the seventies, Schatzman and Moreau undertook to revisit the venerable dynamics of rigid bodies with contact and dry friction in the light of more recent mathematics. One claimed objective was to reach, for the first time, a mathematically consistent formulation of an initial value problem associated with the dynamics. The purpose of this article is to make a review of the today state-of-art concerning not only the formulation, but also the issues of existence and uniqueness of solution.

R É S U M É

À la fin des années 70, Schatzman et Moreau entreprirent de reformuler l'antique dynamique des solides rigides en présence de contact et frottement sec à la lumière de mathématiques plus récentes. Un des objectifs revendiqués était de parvenir, pour la première fois, à la formulation d'un problème d'évolution à partir d'une condition initiale, qui soit mathématiquement cohérent. Le but de cet article est de brosser un état de l'art actuel, concernant non seulement les questions de formulation, mais également d'existence et d'unicité de solution.

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1. Introduction

The dynamics of rigid solids with contact and dry friction conditions is a venerable subject, which was developed mainly in the second half of the nineteenth century and the beginning of the twentieth century to answer some questions raised by engineering. Then, the attention of engineers began to be driven towards elasticity and continuum mechanics, and less attention was paid to frictional contact multibody dynamics. In the seventies, a renewal of interest occurred, mainly driven by the development of numerical modelling in granular dynamics and the control issues associated with robotics. For both concerns, it turned out that the foundations of the venerable theory were not firm enough and that they should be reconsidered in the light of more recent mathematics.

A fundamental impulse was given simultaneously by Michelle Schatzman [1] and Jean-Jacques Moreau [2], who first considered an evolution problem for the configuration $\mathbf{q} : [0, T] \to \mathbb{R}^d$ in the framework of functions whose second derivative (in the distributional sense) is a Radon measure. At that time, the antique point of view of different systems of equations applying to the different phases of motion (without or with active contact) was still prevailing. Their new point of view permitted to formulate, for the first time, a mathematical evolution problem associated with multibody contact dynamics. It also paved the road for the design of efficient strategies for numerical computing and enabled the first investigations on the stability and control issues that are crucial in the analysis of the problems facing frictional contact events in robotics.

The seminal work of Michelle Schatzman and Jean-Jacques Moreau also initiated a series of contributions on the general formulation and the mathematical analysis of the initial value problem associated with multibody contact dynamics. A brief sketch of the history follows.

- The first studies about the formulation were restricted to the model problem of the dynamics of a particle evolving in an admissible region of $\mathbb{R}^d$ bounded by an obstacle. In [1], Michelle Schatzman formulated a consistent evolution problem in the frictionless case, under the additional restriction that the admissible region is convex. She was able to successfully implement a penalty method to prove the existence of a solution for the initial value problem. Her original work was restricted to impacts preserving the kinetic energy (the so-called elastic impact law) and an external force depending only on time. This result was generalized later by Paoli in her PhD thesis [3] to the case of an arbitrary impact law and an external force possibly also depending on current position and velocity. In parallel, an alternative strategy for proving the existence of a solution was designed by Monteiro Marques [4]. He introduced a time-stepping approximation and proved the convergence (of a subsequence) towards a solution. It was restricted to the completely inelastic impact law (zero restitution coefficient), but he was able to relax the convexity assumption of Schatzman on the admissible region. More importantly, he was able to generalize to the case where the contact with the obstacle obeys the Coulomb law of dry friction with a given friction coefficient μ (the frictionless case is recovered by taking $\mu = 0$). One benefit of this new strategy is that it directly suggested an algorithm for numerical computations. The time-stepping approach was further developed by Paoli [5] and [6], who, in particular, extended it to the case of an arbitrary restitution coefficient. Her work, however, is up to now concerned only with the frictionless case.
- It was recognized very early by Michelle Schatzman that issues should be expected with the uniqueness of the solution for the initial value evolution problem. In particular, she exhibited in [1] a striking example of multiple solutions for the unilateral dynamics of the one-degree-of-freedom particle submitted to an external force that is a C^∞ function of time. This issue was further considered by Percivale [7,8], who noticed that the uniqueness of the solution could be recovered in the one-degree-of-freedom problem considered by Schatzman, provided that the given external force was assumed to be not only a C^∞, but also an *analytic* function of time. His work was suggesting that uniqueness could be expected in general, provided a regularity assumption of analyticity on the data. This was proved in full generality for the frictionless problem by Ballard in [9] and [10]. Local uniqueness in the analytic framework was also exploited in this work to design a third alternative strategy (in addition to penalty and time-stepping methods) to prove the existence of the solution. This new strategy turned out to yield more general (except for the additional assumption of analyticity) existence results than those which were available at that time from the penalty and time-stepping methods, encompassing the multi-constraint case with an arbitrary impact law. This strategy of proof was also adapted in [11] and [12] to the dynamics of a point particle with contact conditions and Coulomb friction. It yields a slightly more general (except for the analyticity assumption) existence result than that of Monteiro Marques, and provides, in addition, the uniqueness of the solution from a given initial value.
- In the eighties and the nineties, most of the articles that appeared on the subject of the mathematical formulation of the initial value problem and the issues of existence and uniqueness of solutions were restricted to the model problem of a point particle evolving in $\mathbb{R}^d$, or rather in an admissible region of $\mathbb{R}^d$. There is one noteworthy exception: the seminal article [2] by Jean-Jacques Moreau. In this article, Jean-Jacques Moreau addressed the formulation of the dynamics of a collection of rigid bodies submitted to frictionless unilateral constraints such as the ones arising from the non-interpenetration conditions. The framework is from the beginning that of Lagrange in which the motion is represented as a curve $\mathbf{q}(t)$ in the configuration space, identified with a subset of $\mathbb{R}^d$. In that framework and in accordance with the ideas of Lagrange, the reaction force that appears in the formulation is a *generalized reaction force*. In particular, the detailed distribution of reaction forces in the real world (meaning forces in $\mathbb{R}^3$ from one body onto another) is generally undefined. The existence and uniqueness result of Ballard in [9] and [10] applies to this general framework for frictionless unilateral multibody dynamics, under the assumption of analyticity of the data. It yields existence and