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Mathematical justification of a viscoelastic elliptic membrane problem

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ABSTRACT

We consider a family of linearly viscoelastic elliptic shells, and we use asymptotic analysis to justify that what we have identified as the two-dimensional viscoelastic elliptic membrane problem is an accurate approximation when the thickness of the shell tends to zero. Most noticeable is that the limit problem includes a long-term memory that takes into account the previous history of deformations. We provide convergence results which justify our asymptotic approach.

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1. Introduction

In the last decades, many authors have applied the asymptotic methods in three-dimensional elasticity problems in order to derive new reduced one-dimensional or two-dimensional models and justify the existing ones. A complete theory regarding elastic shells can be found in [1], where models for elliptic membranes, generalized membranes, and flexural shells are presented. It contains a full description of the asymptotic procedure that leads to the corresponding sets of two-dimensional equations. Particularly, the existence and uniqueness of the solution to elastic elliptic membrane shell equations can be found in [2] and in [3]. There, the two-dimensional elastic models are completely justified with convergence theorems.

More recently, in [4], the obstacle problem for an elastic elliptic membrane has been identified and justified as the limit problem for a family of unilateral contact problems of elastic elliptic shells. A large number of actual physical and engineering problems have made it necessary to study models that take into account effects such as hardening and memory of the material. An example of these are the viscoelastic models (see, for example, [5,6]). In some of these models, we can find terms that take into account the history of previous deformations or stresses of the body, known as long-term memory. For a family of shells made of a long-term memory viscoelastic material, we can find in [7–9] the use of asymptotic analysis to justify with convergence results the limit two-dimensional membrane, flexural, and Koiter equations.

In this direction, to our knowledge, in [10] we gave the first steps towards the justification of existing models of viscoelastic shells and finding new ones with the starting point being three-dimensional Kelvin–Voigt viscoelastic shell problems. By using the asymptotic expansion method, we found a rich variety of cases for the limit two-dimensional problems, depending on the geometry of the middle surface, the boundary conditions and the order of the applied forces. The most remarkable feature found was that, from the asymptotic analysis of the three-dimensional problems, a long-term

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memory arose in the two-dimensional limit problems, represented by an integral with respect to the time variable. The aim of this Note is to mathematically justify these equations that we identified in [10] as the viscoelastic elliptic membrane problem, by presenting rigorous convergence results.

2. The three-dimensional linearly viscoelastic shell problem

We denote $\mathbb{S}^d$, where $d = 2, 3$ in practice, the space of second-order symmetric tensors on $\mathbb{R}^d$, while " $\cdot$ " will represent the inner product and $|\cdot|$ the usual norm in $\mathbb{S}^d$ and $\mathbb{R}^d$. In what follows, unless the contrary is explicitly written, we will use summation convention on repeated indices. Moreover, Latin indices $i, j, k, l, \ldots$, take their values in the set $\{1, 2, 3\}$, whereas Greek indices $\alpha, \beta, \sigma, \tau, \ldots$ do it in the set $\{1, 2\}$. Also, we use standard notation for the Lebesgue and Sobolev spaces. Moreover, for a time dependent function u, we denote $\dot{u}$ the first derivative of u with respect to the time variable. Recall that "$\to$" denotes strong convergence, while "$\rightharpoonup$" denotes weak convergence.

Let ω be a domain of $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\gamma = \partial\omega$. Let $\boldsymbol{y} = (y_\alpha)$ be a generic point of its closure $\bar{\omega}$ and let ∂_α denote the partial derivative with respect to y_α.

Let $\boldsymbol{\theta} \in \mathcal{C}^2(\bar{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha(\boldsymbol{y}) := \partial_\alpha \boldsymbol{\theta}(\boldsymbol{y})$ are linearly independent. These vectors form the covariant basis of the tangent plane to the surface $S := \boldsymbol{\theta}(\bar{\omega})$ at the point $\boldsymbol{\theta}(\boldsymbol{y})$. The surface S is uniformly elliptic, in the sense that the two principal radius of curvature are either both positive at all points of S, or both negative at all points of S. We can consider the two vectors $\boldsymbol{a}^\alpha(\boldsymbol{y})$ of the same tangent plane defined by the relations $\boldsymbol{a}^\alpha(\boldsymbol{y}) \cdot \boldsymbol{a}_\beta(\boldsymbol{y}) = \delta^\alpha_\beta$, which constitute the contravariant basis. We define the unit vector, $\boldsymbol{a}_3(\boldsymbol{y}) = \boldsymbol{a}^3(\boldsymbol{y}) := \frac{\boldsymbol{a}_1(\boldsymbol{y}) \wedge \boldsymbol{a}_2(\boldsymbol{y})}{|\boldsymbol{a}_1(\boldsymbol{y}) \wedge \boldsymbol{a}_2(\boldsymbol{y})|}$, normal vector to S at the point $\boldsymbol{\theta}(\boldsymbol{y})$, where $\wedge$ denotes the vector product in $\mathbb{R}^3$.

We can define the first fundamental form, given as a metric tensor, in covariant or contravariant components, respectively, by $a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$, $a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$, the second fundamental form, given as a curvature tensor, in covariant or mixed components, respectively, by $b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha$, $b^\beta_\alpha := a^{\beta\sigma} \cdot b_{\sigma\alpha}$, and the Christoffel symbols of the surface S by $\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha$. The area element along S is $\sqrt{a}\,\mathrm{d}y$, where $a := \det(a_{\alpha\beta})$.

For each $\varepsilon > 0$, we define the three-dimensional domain $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ and its boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon$. We also define the parts of the boundary, $\Gamma^\varepsilon_+ := \omega \times \{\varepsilon\}$, $\Gamma^\varepsilon_- := \omega \times \{-\varepsilon\}$ and $\Gamma^\varepsilon_0 := \gamma \times [-\varepsilon, \varepsilon]$.

Let $\boldsymbol{x}^\varepsilon = (x^\varepsilon_i)$ be a generic point of $\bar{\Omega}^\varepsilon$, and let ∂^ε_i denote the partial derivative with respect to x^ε_i. Note that $x^\varepsilon_\alpha = y_\alpha$ and $\partial^\varepsilon_\alpha = \partial_\alpha$. Let $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ be the mapping defined by

$$\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon) := \boldsymbol{\theta}(\boldsymbol{y}) + x^\varepsilon_3 \boldsymbol{a}_3(\boldsymbol{y}) \ \forall \boldsymbol{x}^\varepsilon = (\boldsymbol{y}, x^\varepsilon_3) = (y_1, y_2, x^\varepsilon_3) \in \bar{\Omega}^\varepsilon \tag{1}$$

If the injective mapping $\boldsymbol{\theta} : \bar{\omega} \to \mathbb{R}^3$ is smooth enough, the mapping $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ is also injective for $\varepsilon > 0$ small enough (see Theorem 3.1-1, [1]). For each ε, $0 < \varepsilon \le \varepsilon_0$ (with ε_0 defined in Theorem 3.1-1, [1]), the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$ is the reference configuration of a viscoelastic shell, with middle surface $S = \boldsymbol{\theta}(\bar{\omega})$ and thickness $2\varepsilon > 0$. Furthermore, for $\varepsilon > 0$, $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon) := \partial^\varepsilon_i \boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$ are linearly independent, and the mapping $\boldsymbol{\Theta} : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ is injective for all ε, $0 < \varepsilon \le \varepsilon_0$, as a consequence of the injectivity of the mapping $\boldsymbol{\theta}$. Hence, the three vectors $\boldsymbol{g}^\varepsilon_i(\boldsymbol{x}^\varepsilon)$ form the covariant basis of the tangent space at the point $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$, and $\boldsymbol{g}^{i,\varepsilon}(\boldsymbol{x}^\varepsilon)$, defined by the relations $\boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^\varepsilon_j = \delta^i_j$, form the contravariant basis at the point $\boldsymbol{\Theta}(\boldsymbol{x}^\varepsilon)$. We define the metric tensor, in covariant or contravariant components, respectively, by $g^\varepsilon_{ij} := \boldsymbol{g}^\varepsilon_i \cdot \boldsymbol{g}^\varepsilon_j$, $g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}$, and the Christoffel symbols by $\Gamma^{p,\varepsilon}_{ij} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial^\varepsilon_i \boldsymbol{g}^\varepsilon_j$.

The volume element in the set $\boldsymbol{\Theta}(\bar{\Omega}^\varepsilon)$ is $\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$, and the surface element in $\boldsymbol{\Theta}(\Gamma^\varepsilon)$ is $\sqrt{g^\varepsilon} d\Gamma^\varepsilon$, where $g^\varepsilon := \det(g^\varepsilon_{ij})$.

Besides, let $T > 0$ be the period of observation and we denote by $u^\varepsilon_i : [0, T] \times \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ the covariant components of the displacement field, i.e. $\boldsymbol{\mathcal{U}}^\varepsilon := u^\varepsilon_i \boldsymbol{g}^{i,\varepsilon} : [0, T] \times \bar{\Omega}^\varepsilon \to \mathbb{R}^3$. For simplicity, we define the vector field $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i) : [0, T] \times \Omega^\varepsilon \to \mathbb{R}^3$, which will denote the vector of unknowns.

We assume that the shell is subjected to a boundary condition of place; in particular, we assume that the displacements field vanishes in $\boldsymbol{\Theta}(\Gamma^\varepsilon_0)$, i.e. on the whole lateral face of the shell.

Let us define the space of admissible unknowns,

$$V(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v^\varepsilon_i) \in [H^1(\Omega^\varepsilon)]^3;\ \boldsymbol{v}^\varepsilon = \boldsymbol{0} \ on \ \Gamma^\varepsilon_0\}$$

This is a real Hilbert space with the induced inner product of $[H^1(\Omega^\varepsilon)]^3$. The corresponding norm is denoted by $||\cdot||_{1,\Omega^\varepsilon}$.

We assume that the body is made of a Kelvin–Voigt viscoelastic material, which is homogeneous and isotropic, so that the material is characterized by its Lamé coefficients $\lambda \ge 0$, $\mu > 0$ and its viscosity coefficients, $\theta \ge 0$, $\rho \ge 0$ (see for instance [5,6]). Under the effect of applied forces, the body is deformed, and we can find that $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i)$ verifies the following variational problem of a three-dimensional viscoelastic shell in curvilinear coordinates:

Problem 2.1. Find $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i) : [0, T] \times \Omega^\varepsilon \to \mathbb{R}^3$ such that

$$\boldsymbol{u}^\varepsilon(t, \cdot) \in V(\Omega^\varepsilon) \ \forall t \in [0, T]$$

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