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Gradient elasticity theory for fiber composites with fibers resistant to extension and flexure

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ABSTRACT

A model for the mechanics of an elastic solid, reinforced with bidirectional fibers is presented in finite plane elastostatics. The fiber's resistance to stretch and flexure are accounted for with variational computations of first and second gradient of deformations, respectively. Within the framework of strain-gradient elasticity, the Euler equation and necessary boundary conditions are formulated. A rigorous derivation of the corresponding linear theory is also developed from which, a complete analytical solution is obtained for small deformations superposed on large. In particular, we assimilate plane bias extension test results indicating that the proposed model successfully predicts smooth transitions of shear strain fields unlike those depicted by the first gradient theory where significant discontinuities are present. The proposed model can serve as an alternative 2D Cosserat theory of non-linear elasticity.

1. Introduction

Mechanics of materials with distinct microstructures have consistently been the subject of intense study (Pipkin, 1979; Spencer, 1972) due to their practical importance in materials science and engineering. It is widely believed that the microstructure of materials, in general, governs the overall mechanical responses (Hahm & Khang, 2010; Monecke, 1989; Moravec & Holecek, 2010; Voigt, 1887). Fiber-reinforced composites are a particular case of such materials where fibers, as the microstructure of the composite, are embedded in a matrix material. In practice, these fibers are often presumed to be densely distributed so as to render the idealization of continuous distribution which further leads to the continuum description of fiber-composites via a homogenization procedure. Within this setting, the composites can be regarded as a special type of anisotropic materials where the response function depends on the first gradient of deformations, typically augmented by the constraints of bulk compressibility or fiber inextensibility. In the latter case, the resulting prediction models are often so constrained that the corresponding deformation fields are essentially kinematically determinate, particularly that arise in fibers (Mulhern, Rogers, & Spencer, 1969; Pipkin & Rogers, 1971). Nonetheless, continuum-based approaches were used widely in the analysis of the mechanics of composite materials for their advantage in the continuum descriptions and the associated mathematical frame work (see, for example, (Mulhern, Rogers, & Spencer, 1967; 1969; Munch, Neff, & Wagner, 2011) and references therein).

The continuum theory, which accounts for the microstructural effects of fibers on elastic materials, has gained renewed attention in recent years (Spencer & Soldatos, 2007; Steigmann, 2012; Steigmann & dell'Isola, 2015). This includes the

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"refinement" of the first-order continuum theory by considering the higher gradient of deformations in an effort to achieve a more detailed characterization of the continua with microstructures. In the case of fiber composites, this means the incorporation of the bending resistance of the fibers into the models of deformations. This is framed in the setting of the nonlinear strain-gradient theory (Koiter, 1964; Toupin, 1964; Truesdell & Noll, 1965) of anisotropic elasticity where the fibers' bending resistance is assigned to the changes in curvature (flexure) of fibers explicitly (Spencer & Soldatos, 2007). The latter is obtained via the computation of the second gradient of deformations in which the fibers are regarded as continuous curves, defined in convected coordinates. Current applications of the general theory are discussed in Maugin and Metrikine (2010); Munch et al. (2011); Neff (2006b); Park and Lakes (1987) , and mathematical aspects of the subject are presented in Fried and Gurtin (2009); Neff (2006a); Park and Gao (2008). A theory for an elastic solid with fiber's resistant to flexure, stretch and twist is developed in Steigmann (2012) under the simplified setting of the constraint Cosserat theory. In addition, authors in dell'Isola, Cuomo, Greco, and Della Corte (2017); dell'Isola, Della Corte, Greco, and Luongo (2016); Turco (2016) discussed second-gradient theory of elasticity for the mechanics of meshed structures and examined shear strain distributions of the meshed structure subjected to the plane bias extension. To this end, authors in Zeidi and Kim (2017a, 2017b, 2018) presented the continuum formulation of fiber-reinforced composites where the bending resistance of fibers is accounted for via the second gradient of deformations. However, the majority of the aforementioned studies were limited in scope, insofar as they considered either inextensible fibers or single family of fibers. Moreover, although recent studies (Cuomo, dell'Isola, & Greco, 2016; dell'Isola et al., 2017; dell'Isola et al., 2016; 2016) reveal that the second-gradient theory accurately predicts the smooth transitions of shear strain fields of meshed structures undergoing bias extensions, compatible results in the case of general fiber composites remain absent from the literature.

In the present work, we develop a continuum model which describes the mechanical responses of an elastic solid reinforced with bidirectional fibers and subjected to finite plane deformations. Thus, it is assumed that the fiber's directors remain in a plane, with no out-of-plane components and that the corresponding deformations and material parameters are independent of the out-of-plane coordinate. The bidirectional fibers are treated as continuously distributed spatial rods of Kirchhoff type such that the kinematics are defined by their position and director fields (Antman, 2005; Dill, 1992; Landau & Lifshitz, 1986). Within this prescription, we propose an energy functional that can accommodate fibers resistant to flexure and extension, and construct the constitutive relations by applying the variational principle on the first and second gradient of deformations, respectively. The corresponding Euler equilibrium equation is also derived by virtue of the virtual work statement. With the Euler equation satisfied, we present a rigorous derivation of the necessary boundary conditions in the case of bidirectional fibers. We then consider a special case of a Neo–Hookian material, reinforced with bidirectional fibers, and subsequently formulate systems of coupled Partial Differential Equations (PDEs) describing the deformations of fiber composites. The solutions of the resulting PDEs are obtained via the Finite Element Analysis (FEA), which demonstrate excellent correspondence with existing theoretical and experimental results (see, for example, (dell'Isola et al., 2017; dell'Isola et al., 2016; Dong & Davies, 2015)).

More importantly, the proposed model assimilates, in the case of bidirectional fiber composites, the plane bias extension test and successfully predicts the smooth transitions of the corresponding shear strain fields, as opposed to the first-gradient theory, where the resulting shear strain appears to be discontinued. In addition, we develop a compatible linear theory within the description of superposed incremental deformations (Ogden, 1984). By employing adapted iterative reduction and eigenfunction expansion methods (Huang & Zhang, 2002; Read, 1993; 1996), a complete analytical solution is obtained, which describes the responses of a bidirectional composite subjected to flexure and extension. The analytical solution is smooth and stable within the entire domain of interest and demonstrates good agreement with the experiment (Dong & Davies, 2015) and FEA results for small deformations superposed on large. Lastly, we note here that the proposed model can serve as an alternative 2D Cosserat theory of nonlinear elasticity (Pietraszkiewicz & Eremeyev, 2009; Reissner, 1987; Toupin, 1964; Truesdell & Noll, 1965).

Throughout the paper, we make use of a number of well-established symbols and conventions such as $\mathbf{A}^T$, $\mathbf{A}^{-1}$, $\mathbf{A}^*$ and $tr(\mathbf{A})$. These are the transpose, the inverse, the cofactor and the trace of a tensor $\mathbf{A}$, respectively. The tensor product of vectors is indicated by interposing the symbol $\otimes$, and the Euclidian inner product of tensors $\mathbf{A}, \mathbf{B}$ is defined by $\mathbf{A}\cdot\mathbf{B} = tr(\mathbf{A}\mathbf{B}^T)$; the associated norm is $|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$. The symbol $|*|$ is also used to denote the usual Euclidian norm of three-vectors. Latin and Greek indices take values in {1, 2} and, when repeated, are summed over their ranges. Lastly, the notation $F_{\mathbf{A}}$ stands for the tensor-valued derivatives of a scalar-valued function $F(\mathbf{A})$.

2. Kinematics and equilibrium equations

Let $\mathbf{L}$ and $\mathbf{M}$ be the unit tangent to the fiber's trajectory in the reference configuration and $\mathbf{l}$ and $\mathbf{m}$ are their counterparts in the deformed configuration. The orientations of particular bidirectional fibers are then given by

$$\lambda = |\eta| = \frac{ds}{dS}, \;\; \mu = |\tau| = \frac{du}{dU} \text{ and } \mathbf{l} = \eta\lambda^{-1}, \;\; \mathbf{m} = \tau\gamma^{-1}, \tag{1}$$

where

$$\mathbf{FL} = \lambda\mathbf{l} \text{ and } \mathbf{FM} = \gamma\mathbf{m}, \tag{2}$$

and $\mathbf{F}$ is the gradient of the deformation function ($\boldsymbol{\chi}(\mathbf{X})$). Eq. (2) can be derived by taking the derivative of $\mathbf{r}(s(S)) = \chi(\mathbf{X}(S))$, with respect to arclength parameters, S, and ultimately, s, upon making the identifications $\mathbf{L} = \frac{d\mathbf{X}}{dS}$ and $\mathbf{l} = \frac{d\chi}{ds}$ and similarly