International Journal of Engineering Science 129 (2018) 63–83

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Significant differences in the mechanical modeling of confined growth predicted by the Lagrangian and Eulerian formulations

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ARTICLE INFO

Article history:
Received 26 June 2017
Revised 6 March 2018
Accepted 13 March 2018

Keywords:
Computational mechanics
Eulerian formulation
Growth
Lagrangian formulation

ABSTRACT

This paper presents an Eulerian formulation of growth for general elastic anisotropic response. The constitutive equations model homeostasis, which is the inelastic process causing the elastic deformation measures to tend towards their homeostatic values. The numerical implementation into ABAQUS using robust, strongly objective integration algorithms for the evolution equations is discussed. Differences between the Eulerian and the Lagrangian multiplicative formulations of growth are examined using examples modeling partially unconfined and confined growth.

1. Introduction

The analysis of growth and remodeling of biological tissues has been an active field in mechanics for decades. Taber (1995) presented a review of the literature related to growth, remodeling and morphogenesis of biological tissues, where he connected growth with addition/removal of mass, remodeling with change in material properties and morphogenesis with change in shape. More recent reviews of growth in living systems can be found in Kuhl (2014) for a solid mechanics formulation, and in Ambrosi et al. (2011), Ateshian and Humphrey (2012) and Sciume et al. (2013) for a mixture theory formulation. Also, it is noted that Humphrey and Rajagopal (2002) proposed a simplified constrained mixture model with no relative motion between the constituents.

The multiplicative form of finite deformation plasticity attributed to Bilby, Bullogh, and Smith (1956), Kröner (1959) and Lee (1969) was first used by Rodriguez, Hoger, and McCulloch (1994) and has become the standard approach to model the inelastic nature of growth. In this model, the total deformation gradient $\mathbf{F}$ is expressed multiplicatively in terms of a growth deformation tensor $\mathbf{F}_g$ from the stress-free reference configuration to an intermediate stress-free configuration and an elastic deformation tensor $\mathbf{F}_e$ from the intermediate configuration to the present deformed configuration, such that

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_g. \tag{1}$$

Recently, Rubin, Safadi, and Jabareen (2015) developed a unified Eulerian theoretical structure for modeling interstitial growth and muscle activation in soft tissues which is based on the ideas of Eckart (1948) and Leonov (1976), who proposed evolution equations directly for elastic deformation measures. Specifically, use is made of the pure separation of dilatation and distortion (Flory, 1961) to propose evolution equations for the elastic dilatation J_e and a symmetric unimodular second order tensor $\mathbf{B}'_e$ $[\det(\mathbf{B}'_e) = 1]$. The theory in Rubin et al. (2015) was developed as a thermomechanical theory treating the

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https://doi.org/10.1016/j.ijengsci.2018.03.008
0020-7225/

tissue as an open system with a rate of mass supply requiring a constitutive equation. A unique feature of this theory is the modeling of homeostasis, which is an inelastic process that causes a tendency for $(J_e, \mathbf{B}'_e)$ to approach their homeostatic values $(J_h, \mathbf{H}')$. A purely mechanical form of this theory was presented in Safadi and Rubin (2017a). Specifically, the constitutive equation for the Cauchy stress $\mathbf{T}$ is specified so that a stress-free state of the tissue is given by

$$\mathbf{T} = 0 \quad \text{for} \quad J_e = 1 \quad \text{and} \quad \mathbf{B}'_e = \mathbf{I}, \tag{2}$$

where $\mathbf{I}$ is the second order identity tensor. In particular, the homeostatic state of the material need not be stress-free

$$\mathbf{T} \neq 0 \quad \text{for} \quad J_e = J_h \quad \text{and} \quad \mathbf{B}'_e = \mathbf{H}', \tag{3}$$

The objectives of the present paper are to generalize the formulation in Safadi and Rubin (2017a) to general elastic anisotropic response and to discuss aspects of the numerical implementation of the model in the commercial finite-element code ABAQUS (2017). A simple example of plane strain axisymmetric deformation of a hollow circular cylindrical tube is used to emphasize the significant differences between the standard Lagrangian formulation (1) and the Eulerian formulation in Rubin et al. (2015) and Safadi and Rubin (2017a). Specifically, it will be shown that modeling homeostasis and the homeostatic state gives control of the stress field in confined growth which need not be present in the standard model. Additional examples presented here related to early cardiac growth processes of the simple heart tube model discussed in Shi, Yao, Xu, and Taber (2014), which used the Lagrangian formulation, demonstrate capabilities of the Eulerian approach to modeling growth in tissues.

An outline of the paper is as follows. Section 2 records generalized equations for growth, Section 3 analyzes stress-free growth, and Section 4 describes the implementation of robust, strongly objective, numerical algorithms in ABAQUS. Section 5 presents a number of examples that emphasize differences in the proposed Eulerian formulation of growth and the standard Lagrangian formulation. Section 6 demonstrates the versatility of the Eulerian formulation in simulating the mechanics of early cardiac morphogenesis, Section 7 presents conclusions and the Appendix summarizes details of some mathematical expressions.

2. Generalized equations for growth

Recall that a material point in the present configuration is located by the position vector $\mathbf{x}$ at time t relative to a fixed point and its velocity $\mathbf{v}$ is given by

$$\mathbf{v} = \dot{\mathbf{x}}, \tag{4}$$

where a superposed $\dot{(\,)}$ denotes material time differentiation. The velocity gradient $\mathbf{L}$ and the total deformation rate tensor $\mathbf{D}$ are defined by

$$\mathbf{L} = \partial \mathbf{v} / \partial \mathbf{x}, \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T). \tag{5}$$

The model in Rubin et al. (2015) was developed as a thermomechanical theory with the tissue considered to be an open system with a rate of mass supply. Here, attention is limited to a purely mechanical theory at constant reference temperature as in Safadi and Rubin (2017a). In this theory, the elastic dilatation J_e is defined as the ratio of the mass density ρ_0 of the tissue in a stress-free state and its current mass density ρ

$$J_e = \frac{\rho_0}{\rho}. \tag{6}$$

Using the modified expression for the rate of mass supply presented in Safadi and Rubin (2017a), the evolution equation for J_e takes the form

$$\frac{\dot{J}_e}{J_e} = \mathbf{D} \cdot \mathbf{I} - \Gamma_m \ln\left(\frac{J_e}{J_h}\right), \quad \Gamma_m \geq 0, \quad J_h > 0, \tag{7}$$

where $\mathbf{A} \cdot \mathbf{B} = \text{tr}(\mathbf{A}\mathbf{B}^T)$ is the inner product between two second order tensors $(\mathbf{A}, \mathbf{B})$, Γ_m is a non-negative function and J_h is the positive homeostatic value of J_e. In addition, the symmetric, unimodular, positive definite, second order elastic distortional deformation tensor $\mathbf{B}'_e$ is determined by the evolution equation

$$\dot{\mathbf{B}}'_e = \mathbf{L}\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}^T - \frac{2}{3}(\mathbf{D} \cdot \mathbf{I})\mathbf{B}'_e - \Gamma\left[\mathbf{B}'_e - \left(\frac{3}{\mathbf{B}'^{-1}_e \cdot \mathbf{H}}\right)\mathbf{H}\right],$$

$$\Gamma \geq 0, \quad \mathbf{H}' = (\det \mathbf{H})^{-1/3}\mathbf{H}, \tag{8}$$

where Γ is a non-negative function and $\mathbf{H}$ is a symmetric positive definite tensor.

The terms associated with (Γ_m, Γ) in (7) and (8) characterize homeostasis, which is the inelastic rate process that causes a tendency for $(J_e, \mathbf{B}'_e)$ to approach their homeostatic values $(J_h, \mathbf{H}')$, respectively. The functions (Γ_m, Γ), which control the rates of homeostasis, and the homeostatic values $(J_h, \mathbf{H}')$ require constitutive equations. Since the functions $(\Gamma_m, \Gamma, J_h, \mathbf{H}')$