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Stress singularity of a notch in a higher-order elastic solid under anti-plane deformation

M.S. Wu

School of Mechanical and Aerospace Engineering, Nanyang Technological University 639798, Singapore

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ABSTRACT

The problem of the stress singularity of a notch in a nonlinear solid under finite anti-plane deformation is investigated using higher-order elasticity. The equilibrium equations are written in terms of the first Piola–Kirchhoff stresses, which are replaced by displacements up to the third-order. The resulting variable-coefficient partial differential equations are solved numerically, subject to vanishing out-of-plane shear tractions on the notch faces. The key results are: (i) the stress exponent characterizing the variation of stress with distance from the notch tip may be positive or negative, implying that the stresses can be non-singular or singular, (ii) the stress exponent becomes more positive with a decrease in the notch angle, (iii) a single dimensionless reduced elastic parameter determines the stress exponent, and (iv) the stress exponent varies with the elastic constants, unlike the case in linear elasticity. Specifically, for a given notch angle the stress exponent becomes more negative with *decrease* in the first Lamé constant λ and the third-order elastic constant n, and with the *increase* in the magnitude of the negative third-order constant m, while it varies non-monotonically with the second Lamé constant (shear modulus) μ.These results have significant implications for notch-like defects in soft solids, e.g., replacement tissues, industrial robots and devices in biomedical applications.

1. Introduction

Analysis of cracks in nonlinear solids undergoing finite deformation is a difficult task. Efforts have been made in the 1970s (Knowles, 1977; Knowles & Sternberg, 1973). These works have focused on the hyperelastic materials characterized by elastic potentials such as the Blatz-Ko and neo-Hookean potentials. A central issue is the asymptotic stress field in the vicinity of a crack, i.e., how the stress varies with distance from the crack tip, as characterized by a stress exponent ω. Long, Krishnan, and Hui (2011) have shown that for an incompressible hyperelasic material under plane stress, ω depends on the strain hardening parameter of the generalized neo-Hookean model. Goriely, Weickenmeier, and Kuhl (2016) have also investigated stress singularities in swelling soft solids. Wu (2017) attempted the problem within the framework of higher-order elasticity, and found that stress singularity may exist and is dependent on the elastic constants.

On the other hand, analysis of notches in nonlinear solids has received virtually no attention. Notches here refer to two surfaces meeting at the apex, each surface making an angle of α, which may be termed the half notch angle, with respect to the plane of symmetry of the notched solid. For a *linear* elastic solid, the asymptotic stress fields in the vicinity of a notch are known, in particular for the case of α=180°, i.e., a sharp crack. In fact, for a linear elastic solid under pure Mode I or Mode III loading, ω varies from −0.5 to 0 as α reduces from 180° to 90°. The corresponding equations for

E-mail address: mmswu@ntu.edu.sg

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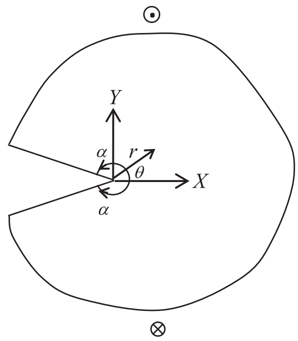

Fig. 1. Schematic of a symmetrical notch of half angle α in a nonlinear elastic solid under finite antiplane deformation. Referential Cartesian coordinates X, Y and polar coordinates r, θ are measured with respect to the notch tip as shown, with Z out of plane.

solving for ω are $\omega \sin 2\alpha + \sin 2\omega\alpha = 0$ for Mode I, and $\cos \omega\alpha = 0$ for Mode III. Under Mode II loading, the corresponding equation is $-\omega \sin 2\alpha + \sin 2\omega\alpha = 0$, and the exponent varies from -0.5 at $\alpha = 180^\circ$ to 0 at $\alpha \sim 128.2^\circ$. Hence, the stresses are non-singular for $\alpha <\sim 128.2^\circ$ in Mode II. This result can be considered anomalous, as it implies that a notch of a given angle in a linear elastic solid may or may not experience singular stresses, depending on the mode of loading. In other words, although a notch always experiences singular stresses under Mode I and Mode III loading, it may or may not experience singular stresses under Mode II, depending on its angle.

A natural question therefore arises regarding the stress exponent in a notched *nonlinear* elastic solid. The present paper addresses this question, starting from the simplest cases and assumptions. Specifically, anti-plane finite deformation of a notched nonlinear elastic solid, characterized by higher-order elasticity, is considered. Such elasticity framework has been developed by, e.g., Murnaghan (1951) and Thurston and Brugger (1964) to take into account crystal anharmonicity when atomic displacements exceed interatomic spacing, if a continuum approximation is adopted. Higher-order elastic constants are used in such theory, and they can be determined via acoustoelastic methods, which essentially rely on the measurement of acoustic wave speeds in stressed solids. For example, the higher-order elastic moduli can be determined from the slope of the ultrasonic wave velocities expressed as functions of uniaxial or hydrostatic stresses applied to the sample (Gennisson et al., 2007). Obviously, other nonlinear constitutive models can be used, such as the neo-Hookean and Blatz-Ko potentials mentioned above. The primary objective of the present work is to determine how the stress exponent ω characterizing the stress variation in the vicinity of a notch varies with the half angle α, assuming the material follows a higher-order elasticity and the notched solid is under finite anti-plane deformation. A nonlinear elastic solid may behave in an even more anomalous manner than its linear counterpart. A classic example is the Poynting effect (Poynting, 1909), in which a material may change its length under pure torsion, or correspondingly, normal stresses are needed to maintain a simple shear deformation (Janmey et al., 2007). Hence, a notched nonlinear elastic solid may also behave quite differently from a purely linear elastic one.

Through experimental measurements and theoretical calculations, Livne, Bouchbinder, Svetlizky, and Fineberg (2010) and Lefranc and Bouchaud (2014) have recently emphasized the importance of elastic nonlinearity and finite deformation in the neighborhood of a crack, even in a macroscopically brittle material. Furthermore, the practical significance of the study of cracks and notches in nonlinear elastic solids cannot be underestimated. The technique of transient elastography has been used to determine the elastic properties of soft tissues and their relation to pathologies (Bercoff et al., 2003). One can consider torn tissues as notches and the investigation of mechanical behavior as a function of elasticity is of critical importance to the integrity of their biological functions. Similarly, replacement tissues and implanted devices are also subjected to tearing and damage, and possibly in a finite deformation setting.

The paper is organized as follows. The problem is defined and the governing equations are presented in Section 2. Numerical results are described in Section 3, followed by a discussion in Section 4. Conclusions are given in Section 5.

2. Problem definition and formulation

Fig. 1 shows a semi-infinite symmetrical notch of half angle α in an infinite solid under anti-plane deformation. Referential coordinates $\mathbf{X} = (X, Y, Z)$ are defined with respect to the notch tip, with the corresponding polar coordinates (r, θ, Z) as shown, where Z is along the out-of-plane direction. The anti-plane deformation is applied along the Z-direction and is defined by:

$$(x, y, z) = (X, Y, Z + w), \tag{1}$$

where $\mathbf{x} = (x, y, z)$ are the current coordinates and w is the displacement along the Z-direction:

$$w = w_1(X, Y) + k w_2(X, Y) + k^2 w_3(X, Y) + ... \tag{2}$$