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On the use of convected coordinate systems in the mechanics of continuous media derived from a QR factorization of F[☆]

Alan D. Freed*, Shahla Zamani

Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843, United States

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ABSTRACT

An oblique, Cartesian, coordinate system arises from the geometry affiliated with a Gram–Schmidt (**QR**) factorization of the deformation gradient **F**, wherein **Q** is a proper orthogonal matrix and **R** is an upper-triangular matrix. Here a cube deforms into a parallelepiped whose edges are oblique and serve as the base vectors for a convected coordinate system. Components for the metric tensor, its dual, and their rates, evaluated in this convected coordinate system, are established for *any* state of deformation. Strains and strain rates are defined and quantified in terms of these metrics and their rates. Quotient laws and their affiliated Jacobians are constructed that govern how vector and tensor fields transform between this oblique coordinate system, where constitutive equations are ideally cast, and the reference, rectangular, Cartesian, coordinate system described in terms of Lagrangian variables, where boundary value problems are solved.

1. Introduction

It was in his 1951 paper entitled “On the use of convected coordinate systems in the mechanics of continuous media” published in the *Proceedings of the Cambridge Philosophical Society* where Arthur Lodge introduced body fields – a formal-

☆ **Dedicated to the memory of Prof. Arthur S. Lodge (1922–2005)**. Arthur Scott Lodge was born in 1922 in Liverpool, England. He received his baccalaureate degree in mathematics and his doctoral degree in theoretical nuclear physics, both from Oxford University. His PhD thesis was conducted under the direction of Maurice Pryce, whose own father-in-law, Max Born, provided inspiration for much of Arthur's subsequent work. He took a position at the British Rayon Research Association in 1948, where he first became interested in the mechanics of materials; in fact, it was there that he was introduced to the emerging discipline of rheology by Karl Weissenberg. In 1961 he moved to a position at the then University of Manchester Institute of Science and Technology (UMIST). In 1964 he published his first monograph, *Elastic Liquids* (Academic Press) (Lodge, 1964), which became a landmark in the field of rheology. It also catalyzed a move to the University of Wisconsin in 1968, where he served as founding Director of the Rheology Research Center. He published his second book, *Body Tensor Fields in Continuum Mechanics, with Applications to Polymer Rheology* (Academic Press) in 1974 (Lodge, 1974), and a third *An Introduction to Elastomer Molecular Network Theory* (Bannatek Press) in 1999 (Lodge, 1999). He retired from the University of Wisconsin in 1991. His work was recognized in many ways, including with the Bingham Medal from The Society of Rheology (1971), the Gold Medal of the British Society of Rheology (1983), and election to the National Academy of Engineering (1992). His name has entered the lexicon of rheology, through the Lodge rubberlike liquid constitutive equation, the Lodge transient network theory, the Lodge–Meissner relation between shear and normal stresses, and the Lodge stressmeter. In all his scientific work he consistently strove for clarity and precision, and was never reluctant to ask basic questions, especially with respect to received wisdom. He also loved the graceful use of the English language, and no author more than P. G. Wodehouse. He had a lifelong passion for classical music, and he played the piano with studied grace and style; he was particularly proud that he could trace the lineage of his piano instructor back to Beethoven. He passed away in 2005, and is survived by his wife of 60 years, Helen Lodge; three children; and seven grandchildren. Written by Tim Lodge, his son.

* Corresponding author.
E-mail address: afreed@tamu.edu (A.D. Freed).

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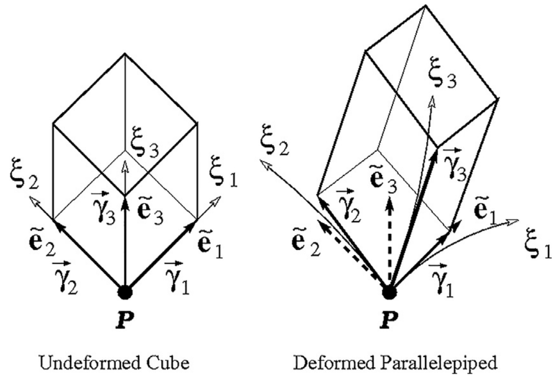

Fig. 1. The left-hand graphic is of a unit cube representing a material element oriented with respect to a set of orthonormal base vectors $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ originating from some particle $\mathcal{P}$. This spatial triad coincides with a set of material lines $\{\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3\}$ that become material curves in the deformed state, the right-hand graphic. Tangents to these material curves $\{\vec{\gamma}_1, \vec{\gamma}_2, \vec{\gamma}_3\}$ describe the edges of a parallelepiped. As volume of the parallelepiped approaches zero, i.e., the volume of particle $\mathcal{P}$, differences between the material curves $\{\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3\}$ and the oblique, Cartesian, tangent vectors $\{\vec{\gamma}_1, \vec{\gamma}_2, \vec{\gamma}_3\}$ become negligible. The oblique, Cartesian, coordinate system becomes coincident with the embedded, curvilinear, coordinate system within a neighborhood surrounding $\mathcal{P}$. Deformation becomes homogeneous as the volume of a material element shrinks to a particle, and the convected coordinate system becomes oblique Cartesian.

ism he made precise in his 1974 book on *Body Tensor Fields in Continuum Mechanics* (Lodge, 1974). Lodge showed that a connection exists between convected space-tensor fields and body-tensor fields, viz., their components are equivalent at that instant when their coordinate axes become coincident (Lodge, 1951). It is at this juncture where we construct our analysis using convected space-coordinate systems derived from the geometry of a parallelepiped generated out of a Gram decomposition of the deformation gradient that, itself, is generated from the motion of a body traveling through space.

Our analysis is based upon the hypothesis: *Deformation is homogeneous at a particle in a continuum.*

Srinivasa (2012) introduced a Gram–Schmidt (**QR**) factorization of the deformation gradient $\mathbf{F}$, which he denoted as $\mathbf{F} = \mathbf{Q}\tilde{\boldsymbol{F}}$, as an alternative to employing the polar decomposition for $\mathbf{F}$ commonly used by the mechanics community. Among the various features that the triangular decomposition of Gram leads to is a coordinate system that is 'nearly' embedded within the material of interest. It is nearly embedded in that a truly embedded coordinate system would move with the body, with its coordinate axes becoming curvilinear, and with each coordinate curve being comprised of the same set of material particles over time, e.g., see Refs. Oldroyd (1950), Lodge (1964), Sokolnikoff (1964), Bird, Armstrong, and Hassager (1987) and Bird, Curtiss, Armstrong, and Hassager (1987).

Practical applications that employ convected coordinate systems have been restricted to the study of certain well-established boundary value problems, e.g., rheological experiments (Bird, Armstrong et al., 1987; Lodge, 1974), whose coordinate systems have global reach. The challenge has been to quantify the relevant, convected, tensor fields for any arbitrary state of deformation, e.g., as they would arise in a finite element analysis. The objective of this paper is to take a step towards resolving this long-standing challenge. This is achieved by constructing a convected coordinate system with local reach, i.e., being applicable within a neighborhood surrounding a particle in some body of interest.

The partially embedded coordinate system arrived at in our analysis is oblique Cartesian. It convects with the motion, but only within a neighborhood surrounding a particle. The coordinate axes defining this system are comprised of tangents to an embedded curvilinear triad whose origin is located at the particle $\mathcal{P}$ whereat deformation gradient $\mathbf{F}(\mathcal{P})$ is evaluated, see Fig. 1. The rectangular, Cartesian, coordinate system associated with a Gram–Schmidt factorization of $\mathbf{F}$ has a 1-coordinate direction that remains tangent to the 1-embedded curve at the particle where $\mathbf{F}$ is evaluated; plus, it has a 12-coordinate plane that remains tangent to the 12-embedded surface at this particle, see Srinivasa (2012, Fig. 1) and Freed and Srinivasa (2015, Fig. 1). It is in this rectangular, Cartesian, coordinate system, rotated by $\mathbf{Q}$ out of a Lagrangian coordinate system, wherein an experimentalist can uniquely and unambiguously measure all six physical components of distortion $\tilde{\mathbf{F}}$, hence its namesake: the experimentor's frame of reference.

A convected metric with local reach is derived in this paper that describes a homogeneous state of deformation in terms of an oblique, Cartesian, coordinate system, provided one knows components of the deformation gradient. The larger challenge of providing a convected metric with global reach to describe non-homogeneous motions in terms of a single, curvilinear, coordinate system, valid for arbitrary states of deformation, remains an unsolved problem in mechanics. That description, if and when it is solved, will reduce to our description within a neighborhood surrounding their coordinate origin.

This paper derives the covariant and contravariant base-vectors, metrics, strains, and their differential rates in this convected coordinate system, in the sense of Lodge (1951, 1956, 1958, 1964, 1972, 1974, 1975, 1984, 1999). These fields are pulled back into the Lagrangian configuration for their use in analysis. These fields describe the geometry of distortion, namely, that of a parallelepiped. (Stretch describes the geometry of an ellipsoid.) Distortion $\tilde{\mathbf{F}}$ is the upper-triangular contri-