International Journal of Engineering Science 124 (2018) 16–23

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

On the anisotropy of cracked solids

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ARTICLE INFO

Article history:
Received 25 October 2017
Accepted 25 November 2017

Keywords:
Effective elastic properties
Cracks
Interactions
Anisotropy
Orthotropy

ABSTRACT

We consider the effective elastic properties of cracked solids, and verify the hypothesis that the effect of crack interactions on the overall *anisotropy* – its type and orientation – is negligible (even though the effect on the overall elastic constants may be strong), provided crack centers are located randomly. This hypothesis is confirmed by computational studies on large number of 2-D crack arrays of high crack density (up to 0.8) that are realizations of several orientation distributions. Therefore, the anisotropy can be accurately determined analytically in the non-interaction approximation (NIA). Since the effective elastic properties possess the orthotropic symmetry in the NIA (for any orientation distribution of cracks, including cases when, *geometrically*, the crack orientation pattern does not have this symmetry), the orthotropy of cracked solids is not affected by interactions.

1. Introduction. The hypothesis to be verified

At low-to-moderate crack densities, the effect of crack interactions on the effective elastic properties is weak (weaker than one may expect). The reason is that, although the effect of interactions on *local* quantities such as stress intensity factors (SIFs) may be strong, the opposite effects of shielding and amplification (in arrays of cracks with random mutual positions) largely cancel one another, as directly seen from computations of Grechka and Kachanov (2006). This can also be explained by applying the internal variables technique of Rice (1975) to cracked solids: the effect of interactions on contributions of cracks to the effective properties is substantially weaker than their effect on local quantities such as SIFs (see Kachanov & Sevostianov, 2012).

Therefore, the non-interaction approximation (NIA), as applied to cracked solids, has larger-than-expected range of applicability – provided that the proper version of the NIA is used: compliances, and not stiffnesses, are linear in crack density, and the NIA is not confused with its linearized version, the "dilute limit" (see the discussion of Sevostianov & Kachanov, 2012). However, at high crack densities, the effect of interactions on the overall properties (the difference with the NIA results) becomes substantial – particularly in 2-D geometries (see, for example, computations of Kushch, Sevostianov, & Mishnaevsky, 2009 and experimental data on microcracked ceramics of Bruno & Kachanov, 2016).

The *anisotropy* of the effective properties caused by non-random orientations of cracks is another factor of importance (its knowledge may be needed, for example, for proper interpretation of various wavespeed data). The question arises, *whether the two factors – the reduction of stiffness and the anisotropy – can be separated*. We examine the hypothesis that the anisotropy can be accurately determined in the NIA, in spite of strong interactions. We assume that the anisotropy

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https://doi.org/10.1016/j.ijengsci.2017.11.023

is solely due to orientation distribution of cracks (crack centers do not form regular spatial patterns, such as lattice, that possess symmetries of their own).

This hypothesis is examined by computational studies of 2-D crack arrays of high crack density that have different orientation distributions. The applicability of the findings to 3-D crack geometries is discussed in the last Section of the work.

2. Background results. Anisotropy of cracked solids in the non-interaction approximation

In the NIA, the effective elastic properties of an isotropic matrix with certain distribution of cracks always possess the orthotropic symmetry – even if, *geometrically*, the orientation distribution of cracks does not have this symmetry (for example, in the case of two families of parallel cracks oriented at arbitrary angle to one another), see papers Kachanov (1980) and Kachanov (1992). The reason for this, somewhat counterintuitive, fact is that a set of 2-D non-interacting cracks can be fully characterized – from the viewpoint of the effective elastic properties – by symmetric second-rank crack density tensor (its 2-D version)

$$\boldsymbol{\alpha} = (1/A)\sum_{\mathrm{k}} \left(\mathrm{a}^2 \boldsymbol{nn}\right)^{(k)} \quad in\,components, \;\; \alpha_{ij} = (1/A)\sum_{k} \left(\mathrm{a}^2 n_i n_j\right)^{(k)}, \tag{2.1}$$

where A is the area of averaging domain (2-D RVE) and $2a^{(k)}$ is the length of kth crack. Its trace $\rho \equiv \alpha_{ii} = (1/A)\sum \mathrm{a}_{\mathrm{k}}^2$ is a 2-D version of the scalar crack density parameter introduced (in 3-D case) by Bristow (1960) for randomly oriented cracks; hence $\boldsymbol{\alpha}$ is a tensor generalization of ρ that accounts for crack orientations.

Indeed, the change in elastic compliances due to cracks in the 2-D case, in the NIA, is expressed in terms of $\boldsymbol{\alpha}$. The effective compliance tensor is represented as $S_{ijkl} = S^0_{ijkl} + \Delta S_{ijkl}$, where S^0_{ijkl} are compliances of the isotropic bulk material

$$S^0_{ijkl} = \frac{1+\nu'_0}{2E'_0}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{\nu'_0}{E'_0}\delta_{ij}\delta_{kl} \tag{2.2}$$

and ΔS_{ijkl} are changes due to cracks

$$\Delta\, S_{ijkl} = \frac{\pi}{E'_0}\frac{1}{4}\left(\delta_{ik}\alpha_{jl} + \delta_{jl}\alpha_{ik} + \delta_{jk}\alpha_{il} + \delta_{il}\alpha_{jk}\right). \tag{2.3}$$

Equivalently, the result (2.3) can be expressed in terms of the change of the elastic potential due to cracks:

$$\Delta f = \left(\pi/E'_0\right)(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}):\boldsymbol{\alpha} = \left(\pi/E'_0\right)\sigma_{ij}\sigma_{jk}\alpha_{ik}. \tag{2.3a}$$

Hereafter, E'_0 and ν'_0 are 2-D Young's modulus and Poisson's ratio: E'_0 equals 3-D modulus E_0 for plane stress and $E_0/(1-\nu_0^2)$ for plane strain and Poisson's ratio $\nu'_0 = \nu_0$ for plane stress and $\nu'_0 = \nu_0/(1-\nu_0)$ for plane strain. In the present work, the 3-D Poisson's ratio $\nu_0 = 1/4$ will be assumed, implying $\nu'_0 = 1/3$ in the case of plane strain.

Since $\boldsymbol{\alpha}$ is symmetric second-rank tensor, the effective elastic properties are always orthotropic, the orthotropy axes being coaxial with the principal axes of $\boldsymbol{\alpha}$. We emphasize that the exact orthotropic symmetry holds only in the NIA. Indeed, for interacting cracks, tensor $\boldsymbol{\alpha}$ becomes, strictly speaking, inadequate as crack density parameter: it contains no information on mutual positions of cracks that become relevant for interacting cracks.

Remark. The same comment, that the concentration parameters that do not reflect the mutual positions of inhomogeneities, are, strictly speaking, inadequate at finite concentrations, applies to other commonly used parameters, such as scalar crack density ρ, or volume fraction, for other types of inhomogeneities. Although these parameters may distort the actual contributions of individual inhomogeneities to the effective properties, there is no simple alternative (short of solving the interaction problem). Referring to Kachanov and Sevostianov (2005) for further discussion, we focus here on the specific issue of *anisotropy* – whether it can be predicted by tensor $\boldsymbol{\alpha}$.

We examine the issue of the orthotropic symmetry for *interacting* cracks. Deviations from the orthotropic symmetry will be measured by the dimensionless Euclidean norm

$$\delta = \left\|\boldsymbol{S}^{ortho} - \boldsymbol{S}^{actual}\right\| \Big/ \left\|\boldsymbol{S}^{actual}\right\| = \sqrt{\left(S^{ortho}_{ijkl} - S^{actual}_{ijkl}\right)\left(S^{ortho}_{ijkl} - S^{actual}_{ijkl}\right)}\Big/\sqrt{S^{actual}_{ijkl}S^{actual}_{ijkl}}, \tag{2.5}$$

where $\boldsymbol{S}^{ortho}$ is the "best-fit" orthotropic compliance tensor.

The concept of approximate and best-fit elastic symmetries was introduced by Fedorov (1968) where the best-fit isotropic approximation of elastic anisotropies was given. It was further developed, in the context of geophysics applications, by Arts, Rasolofosaon, and Zinsner (1996). For a systematic treatment of the concept and further results, see Sevostianov and Kachanov (2008); they found, in particular, that the best-fit orthotropic approximation of a given non-orthotropic tensor S_{ijkl} is given simply by setting the non-orthotropic compliances (such as S_{1112}, S_{2212}) in the axes of the best-fit orthotropy equal to zero.

Remark. In the 3-D case (circular cracks of radii a_k) – that is not considered here – crack density tensor $\boldsymbol{\alpha} = (1/V)\sum_{\mathrm{k}}\left(\mathrm{a}^3\boldsymbol{nn}\right)^{(k)}$has to be supplemented by fourth-rank tensor $\boldsymbol{\beta} = (1/V)\sum_{\mathrm{k}}\left(\mathrm{a}^3\boldsymbol{nnnn}\right)^{(k)}$ that, however, plays a secondary role (it enters $\Delta\, S_{ijkl}$ with multiplier $\nu_0/2$ that is substantially smaller than the overall coefficient of 1 at the $\boldsymbol{\alpha}$-terms):

$$\Delta S_{ijkl} = \frac{32\left(1-\nu_0^2\right)}{3(2-\nu_0)E_0}\left[\frac{1}{4}\left(\delta_{ik}\alpha_{jl} + \delta_{jl}\alpha_{ik} + \delta_{jk}\alpha_{il} + \delta_{il}\alpha_{jk}\right) - \frac{\nu_0}{2}\beta_{ijkl}\right]. \tag{2.6}$$