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On the determination of semi-inverse solutions of nonlinear Cauchy elasticity: The *not so simple* case of anti-plane shear

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This paper is dedicated to Mike Hayes with esteem and gratitude.

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ABSTRACT

We provide a systematic and complete analysis of the overdetermined problem that one obtains while considering the balance equations of unconstrained isotropic nonlinear Cauchy elastic bodies undergoing anti-plane shear deformations.

1. Introduction

In the 1951 survey paper by Neményi (1951) concerning semi-inverse methods in mechanics of continua there is no mention of the existence of exact solutions for nonlinear classical Cauchy elasticity despite the fact that the first paper where exact solutions are provided is dated 1915 (Armanni, 1915) and Rivlin's elegant excursion in elasticity is dated 1948 (Rivlin, 1948). A first list of exact solutions for nonlinear elasticity appears in 1952 in the celebrated paper by Truesdell (1952) regarding the foundations of the mechanics of Continua. In 1956 in the survey paper by Doyle and Ericksen (1956) the list of exact solutions is considered in some details and in the 1960 book by Adkins and Green (1960) dedicated to large elastic deformations an entire chapter is devoted to *General Solutions of Problems* where a careful survey of some exact solutions up to 1960 for some elastic materials is presented. Nearly any book appearing after 1960 on subjects cognate to nonlinear continuum mechanics contains at least a chapter on exact solutions in nonlinear elasticity, but despite this evidence, and in some contrast to fluid mechanics,[1] an adequately complete survey dedicated to exact solutions in nonlinear elasticity is still missing. Up to now, the (partial) review papers on this subject that we may record are (Hill, 2001a, 2001b) dedicated only to Varga and Neo-Hookean materials, respectively, and Horgan (1995) where only anti-plane shear deformations are considered. The reason is probably to be found in the fact that the only methodological framework at our disposal to search for exact solutions of the highly nonlinear field equations of nonlinear elasticity is the semi-inverse method based on *ad hoc* assumptions. Recently, Rajagopal (2003, 2006) has generalized the class of bodies that are elastic in that they are incapable of dissipation.

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[1] The exact solutions of the Navier–Stokes equations have been presented in a coherent manner, in part by classifying solutions via their temporal and geometric constraints, in books (Berker, 1963; Drazin & Riley, 2006) and in the survey papers (Wang, 1989, 1991). For non-Newtonian fluids the situation is similar to the one we have for nonlinear elasticity.

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These bodies defined through implicit relations between the deformation gradient and the stress includes Cauchy elastic bodies as a special sub-class. General elastic materials are a perfect example that vividly illustrates the complexity associated with the semi-inverse method. In the classical semi-inverse method one assumes a form for the velocity or displacement field, and in other field theories, the form for what is regarded as the independent field variable. However, when considering implicit theories, one has to make special choices for all the variables that are related to one-another implicitly. In this article, we shall only concern ourselves with the mechanics of Cauchy elastic bodies.

The program proposed by Ericksen (1977) to relate semi-inverse methods of Continuum Mechanics to group analysis of differential equations (Olver, 1993) has been explored only with regard to simple but popular kind of groups and therefore it is still in an immature stage of development. The main problem is that a semi-inverse method reduces the basic general field equations of a elasticity theory to an overdetermined system of differential equations and the formal compatibility[2] of this overdetermined system is a serious obstruction to the realization of a general and comprehensive methodology of approach to the problem.

The aim of this paper is to give a complete and systematic resolution of the compatibility problem arising in nonlinear isotropic elastostatics with the application of the semi-inverse method within the framework of the anti-plane shear deformation. It is well known that anti-plane shear is one of the simplest classes of deformations of solids and that problems involving deformations of this kind have been often helpful in the study of qualitative effects, whose analogues, in more elaborate deformations are much less accessible because of the tremendous technical complexities that may accompany a nonlinear theory. We reconsider the determining equations for this class of deformations as a playground to understand the underpinnings of the formal problem of compatibility associated with semi-inverse problems in nonlinear elastostatics.

Towards this end we start fixing some of the basic terminology using the following definitions:

- A *controllable solution* is a deformation which satisfies the equilibrium equations of nonlinear elasticity given a specific body force, that is the deformation is controlled by means of the application of surfaces traction alone.
- A *general solution* is a controllable solution which is admissible for any constitutive equation in a given class of materials.
- A controllable solution that is the same for all materials in a given class is an *universal solution*. Therefore, an universal solution is a general solution with the same kinematics for all materials in the given class.
- A *partial solution* is a controllable solution with a constitutive dependent geometry (or kinematics) that is possible only for a subset of the given class of materials.

Universal solutions for incompressible and compressible (unconstrained) isotropic elastic materials have been determined in Ericksen (1954, 1955). These solutions are precious from the standpoint of attempting to determine experimentally the form of the strain energy density function of an elastic material and a detailed review on this topic is provided by Saccomandi (2001).

General solutions are not rare in nonlinear elasticity. It may happens that, using the semi-inverse method, a given ansätz on the deformation field reduces the balance equations of nonlinear elasticity to a well determined system of differential equations. This system of equations admits formally, for any reasonable constitutive choice, a family of solutions. The geometry (or kinematics) of such controllable solutions depends on the specific constitutive equation. Example of such solutions are the rectilinear, axial, azimuthal or helical shear deformations for isotropic incompressible materials (Ogden, 1984).

An example of a partial solution is the anti-plane shear deformation for isotropic incompressible materials. This is a deformation admissible only for special choices of the constitutive functions within the framework of this class of materials. A list of such solutions may be found in books (Adkins & Green, 1960; Antman, 1995; Green & Zerna, 1954; Ogden, 1984).

We point that we are using the term *class of materials* with some naivety. Usually by a class of materials we are denoting all materials with specified structural properties, e.g. all isotropic materials or all incompressible isotropic materials. In other frameworks the class of materials may be identified via a more restrictive definition, e.g. all incompressible isotropic materials whose strain-energy density functions depends on the first principal invariant of a strain measure.

Ericksen's theorems contained in Ericksen (1954, 1955) about universal solutions in finite isotropic elasticity has had a profound influence in the development of the field. The fact that the only universal solutions for unconstrained materials are the homogeneous deformations and that for incompressible materials only the five families of universal deformations are possible, seems to have produced for several years the *false* impression that the determination of exact solutions in finite elasticity (and especially for unconstrained materials) is hopeless. During the seventies very few papers were devoted to search for exact solutions in nonlinear Cauchy elasticity. A certain amount of effort has been devoted to close in a definitive manner the search for universal solutions for incompressible isotropic elastic materials characterized by constant strain invariants (see Saccomandi (2001) for the details), but a very short list of papers devoted to general and/or partial solutions have been recorded. A notable exception to this state of affairs has been the important activity by Ogden and co-workers summarized in Ogden (1984) and the papers by Knowles on anti-plane shear deformations (Knowles, 1976, 1977).

[2] We point out that we are not speaking of the compatibility conditions for the right Cauchy–Green strain tensor, but of the integrability conditions for a general overdetermined system of partial differential equations.