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Modeling fracture in the context of a strain-limiting theory of elasticity: A single plane-strain crack

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ABSTRACT

Implicit constitutive relations afford a logically consistent framework for formulating strain-limiting, nonlinear elastic constitutive models utilizing the classical linearized strain tensor. This is in marked contrast to traditional Cauchy or Green elastic formulations which give rise to customary linear elasticity in the infinitesimal strain limit. Within this strain-limiting elastic constitutive setting, the present paper investigates the asymptotic behavior of the stress field at the tip of a straight plane-strain fracture. It is shown that within the general class of crack-tip asymptotic expansions considered, the only cases satisfying the required boundary conditions correspond to bounded stresses. This is in agreement with previous results for the corresponding anti-plane shear fracture problem.

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1. Introduction

In [Rajagopal and Walton \(2011\)](#), a study of brittle fracture was initiated in the context of strain limiting theories of elasticity. The motivation for studying brittle fracture in such a setting was multifaceted. Firstly, classical fracture mechanics within the linearized theory of elasticity has the well-known internal logical inconsistency of predicting strain singularities within a model of material behavior predicated upon infinitesimal strains ([Broberg, 1999](#)). Secondly, there is solid experimental evidence that many elastic-like (non-dissipative) materials exhibit significant nonlinear response even well within the small strain regime ([Li, Morris, Nagasako, Kuramoto, & Chrzan, 2007](#); [Rajagopal, 2013](#); [Saito et al., 2003](#); [Talling, Dashwood, Jackson, Kuramoto, & Dye, 2008](#); [Withey et al., 2008](#); [Zhang, Li, Jia, Hao, & Yang, 2009](#)). Moreover, Rajagopal and co-authors have demonstrated how strain limiting theories of elasticity, as a special case of the more general class of implicit elastic constitutive relations, provide a convenient framework for deriving nonlinear elastic constitutive models based upon the traditional linearized strain tensor ([Rajagopal, 2003, 2007, 2010, 2011](#); [Rajagopal & Srinivasa, 2009](#)). This is in marked contrast to the infinitesimal strain limit derived from classical Cauchy or Green formulations of elasticity¹ which must necessarily result in a linear constitutive response.

The problem considered in [Rajagopal and Walton \(2011\)](#) was that of a single, anti-plane shear crack in an elastic material body with a strain-limiting constitutive model of the form:

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¹ [Carroll \(2009\)](#) has recently shown that a Cauchy elastic material that is not Green elastic would be a source of infinite energy. Such an eventuality was recognized by [Green \(1837, 1839\)](#) in his seminal papers.

$$\epsilon = \phi(\beta|\tau)|\tau \quad (1)$$

in which $\beta > 0$, the stress, τ , has been nondimensionalized by the linear elastic shear modulus,

$$\epsilon := \nabla w_3 = \begin{pmatrix} \epsilon_{13} \\ \epsilon_{23} \end{pmatrix} \quad \text{and} \quad \tau := \begin{pmatrix} \tau_{13} \\ \tau_{23} \end{pmatrix}$$

and $\phi(r)$ is a decreasing function satisfying $\lim_{r \rightarrow \infty} r\phi(r) = 1$. It follows that the strain, ϵ , satisfies

$$|\epsilon| < \frac{1}{\beta} \quad (2)$$

and hence is infinitesimal provided $\beta \gg 1$. It was shown in [Rajagopal and Walton \(2011\)](#) that the model (1) is invertible with the equivalent Cauchy elastic formulation:

$$\tau = \psi(\beta|\epsilon)|\epsilon \quad (3)$$

with

$$\psi(r) := \frac{1}{\phi(\xi^{-1}(r))}. \quad (4)$$

Most of the analysis presented in [Rajagopal and Walton \(2011\)](#) was for the special case:

$$\phi(r) = \frac{1}{1+r} \quad \text{and} \quad \psi(r) = \frac{1}{1-r}. \quad (5)$$

For the anti-plane strain fracture problem considered ([Rajagopal & Walton, 2011](#)), no crack-tip strain singularity can occur due to use of the strain limited constitutive model. Of interest in [Rajagopal and Walton \(2011\)](#), was the question of whether or not there can be a crack-tip stress singularity. It was observed in [Rajagopal and Walton \(2011\)](#) that this question is more easily studied from the constitutive formulation (1) rather than the equivalent Cauchy formulation (3). To that end, it proved advantageous to introduce a stress potential Φ satisfying:

$$\tau_{13} = \partial_{x_2} \Phi \quad \text{and} \quad \tau_{23} = -\partial_{x_1} \Phi, \quad (6)$$

which guarantees automatic satisfaction of the equilibrium equation. The strain compatibility equation, $\epsilon_{13,2} = \epsilon_{23,1}$, results in the governing second-order partial differential equation:

$$0 = \Delta \Phi - \frac{\beta\phi(\beta|\tau|)}{|\tau|} \left((\partial_{x_2} \Phi)^2 \partial_{x_2}^2 \Phi + (\partial_{x_1} \Phi)^2 \partial_{x_1}^2 \Phi - 2(\partial_{x_2} \Phi)(\partial_{x_1} \Phi) \partial_{x_1 x_2}^2 \Phi \right) \quad (7)$$

with associated boundary conditions: $\Phi(x_1, 0+) = 0$ for $x_1 < 0$ and $\partial_{x_2} \Phi(x_1, 0+) = 0$ for $x_1 > 0$. Asymptotic analysis at the crack tip could throw light on the question of whether there is a crack-tip singularity in stress. While the nonlinear boundary value problem (7) precludes separable solutions and linear superposition, it was shown in [Rajagopal and Walton \(2011\)](#), that it does admit crack-tip asymptotic representations of the form (relative to a polar coordinate system centered at the crack tip):

$$\Phi(r, \theta) \sim \sum_{j,k=0}^{\infty} r^{\alpha j+k} \hat{\Phi}_{jk}(\theta), \quad \text{as } r \rightarrow 0+. \quad (8)$$

It was shown in [Rajagopal and Walton \(2011\)](#) that only for $\alpha > 1$ does the asymptotic series (8) satisfy the required boundary conditions, and hence that the boundary value problem (7) does not admit crack-tip stress singularities within the general class (8). In a rather different direction, it was shown in [Bulíček, Málek, Rajagopal, and Walton \(submitted for publication\)](#) for a family of strain-limiting constitutive models that includes (3) and (15), that solution to the boundary value problem (7) with accompanying boundary conditions exists in an appropriate Sobolev space.

The present contribution investigates whether the above anti-plane shear results described above hold also for the corresponding plane-strain fracture problem. While the crack-tip asymptotic analysis of a plane-strain crack in the strain-limiting constitutive setting is considerably more complicated than the anti-plane shear problem, it is shown that similar results obtain, namely within the class of crack-tip asymptotic series considered, no cases that would result in singular stresses satisfy the required boundary conditions. However, within the class of asymptotic series investigated, there are cases corresponding to bounded stresses for the which the boundary conditions are satisfied. It should be noted that [Tarantino \(1996\)](#) has given a detailed asymptotic analysis, similar to that presented here, of the crack-tip stress and strain fields for the corresponding plane stress fracture problem. However, Tarantino considered a conventional hyperelastic constitutive model for which neither stress nor strain are constitutively limited. Thus, his crack-tip asymptotic analysis reveals the expected stress and strain singularities. It also bears noting that [Knowles \(1977\)](#) gave an asymptotic analysis for the anti-plane shear crack problem for a class of hyperelastic models that includes as a special case, the *stress limiting* constitutive relation (using the notation of [Rajagopal & Walton \(2011\)](#) rather than [Knowles \(1977\)](#)):

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