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Analysis of the effects of rough surfaces in compressible thin film flow by homogenization

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ABSTRACT

A classical problem in lubrication theory is to predict the pressure distribution in a thin fluid film between two surfaces which are in relative motion. If one of the surfaces is rough, then the distance between the surfaces is rapidly oscillating. This leads to that the governing Reynolds partial differential equation involves rapidly oscillating coefficients. The branch in mathematics which considers such types of equations is known as homogenization. In this paper we study the effects of surface roughness for a special type of compressible fluid. In particular, we derive homogenization results connected to the friction force and the load carrying capacity.

1. Introduction

An essential issue in lubrication theory is to describe the flow behavior between two close surfaces which are in relative motion. This type of flow take place in for example bearings, hip joints, gearboxes etc. The main unknown is the pressure in the fluid. When the pressure is found it is possible to compute other fundamental quantities as friction force and load carrying capacity. The friction force gives information about which force that has to be applied to keep the surfaces in relative motion and the load carrying capacity is the load carried by the surfaces. In this paper we will consider the effects of surface roughness under the assumption that the fluid has constant bulk modulus (see (2) below).

Assume that we have two surfaces, one lower and one upper. The lower surface, S_l, is smooth. For simplicity we let S_l lie in the x_1, x_2-plane. Indeed, let Ω be an open bounded subset of $\mathbb{R}^2$ and $x = (x_1, x_2) \in \Omega$ then S_l is described by $\{(x,0) \in \mathbb{R}^3 : x \in \Omega\}$. Let us now turn to the description of the upper surface, which contrary to the lower surface includes surface roughness. In order to describe the upper surface we introduce an auxiliary function $0 < \alpha_1 \leqslant h(x, y) \leqslant \alpha_2$, which is Y-periodic in y. The upper surface, S_u^ε, is described, in terms of the auxiliary function h, as $\{(x, h_\varepsilon(x)) \in \mathbb{R}^3 : x \in \Omega\}$, where $h_\varepsilon(x) = h(x, x/\varepsilon)$, $\varepsilon > 0$. An important example, from an application point of view, is when h is of the form $h(x, y) = h_0(x) + h_r(y)$. This means that h_0 describes the global geometry, h_r represents an evenly distributed roughness on the upper surface and $\varepsilon > 0$ is a parameter which describes the fineness of the roughness. Note that h_ε is the distance between the surfaces.

Assume that the upper surface is stationary and that the lower surface is moving in the x_1 direction with the speed v. If a fluid with constant viscosity, η, and a pressure dependent density, ρ, occupies the region between the surfaces, then the pressure, p_ε, in the fluid, due to the relative motion of the surfaces, is often modeled by the Reynolds equation

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$$\operatorname{div}\left(\frac{h_\varepsilon^3(x)}{12\eta}\rho(p_\varepsilon(x))\nabla p_\varepsilon(x)\right)=\frac{v}{2}\frac{\partial}{\partial x_1}(\rho(p_\varepsilon)h_\varepsilon),\quad \text{in } \Omega. \tag{1}$$

Without loss of generality we assume that $p_\varepsilon = p_a$ at the boundary (p_a is a constant ambient pressure). For a derivation of Reynolds equation see e.g. the original work (Reynolds, 1886) or the book (Hamrock, 1994). A mathematical proof of the transition between the Stokes equations and the Reynolds equation can be found in Bayada and Chambat (1986).

The main focus of this paper is to study the effects of surface roughness in the case when the relation between the density and the pressure is of the form

$$\rho(p_\varepsilon)=\rho_a e^{(p_\varepsilon-p_a)/\beta}. \tag{2}$$

Here the constant ρ_a is the density at the ambient pressure and β is a positive constant (bulk modulus). We remark that this relation is equivalent to the commonly used assumption that the lubricant has constant bulk modulus β, (see e.g. Elrod, 1981 & Fredrik et al., Fredrik, Andreas, Roland, & Sergei, 2007).

Due to the special form of the relation (2) it is possible to transform the nonlinear Eq. (1) into a linear equation. Indeed, define the function w_ε as

$$w_\varepsilon(x)=\rho(p_\varepsilon(x)). \tag{3}$$

Then

$$\nabla w_\varepsilon=\rho'(p_\varepsilon)\nabla p_\varepsilon=\rho_a\beta^{-1}e^{(p_\varepsilon-p_a)/\beta}\nabla p_\varepsilon=\beta^{-1}\rho(p_\varepsilon)\nabla p_\varepsilon$$

and the Eq. (1) is converted to the linear equation

$$\operatorname{div}\left(h_\varepsilon^3\nabla w_\varepsilon\right)=\lambda\frac{\partial}{\partial x_1}(w_\varepsilon h_\varepsilon),\quad \text{in } \Omega, \tag{4}$$

where $\lambda = 6\eta v/\beta$ and $w_\varepsilon = \rho_a$ on the boundary. In order to get zero boundary condition we introduce $u_\varepsilon = w_\varepsilon - \rho_a$. In this notation the Eq. (4) is

$$\operatorname{div}\left(h_\varepsilon^3\nabla u_\varepsilon\right)-\lambda\frac{\partial}{\partial x_1}(h_\varepsilon u_\varepsilon)=\lambda\rho_a\frac{\partial h_\varepsilon}{\partial x_1},\quad \text{in } \Omega, \tag{5}$$

where $u_\varepsilon = 0$ on the boundary.

The friction force $F_\varepsilon = (F_1^\varepsilon, F_2^\varepsilon)$ at the surface $x_3 = 0$ is given by

$$F_\varepsilon=\int_\Omega \frac{1}{2}h_\varepsilon(x)\nabla p_\varepsilon(x)+\frac{\eta v}{h_\varepsilon(x)}\,dx. \tag{6}$$

The force which is needed to run the surface (e.g. a bearing) is F_1^ε. Using the transformation (3)

$$\nabla p_\varepsilon=\frac{\beta}{w_\varepsilon}\nabla w_\varepsilon=\frac{\beta}{(u_\varepsilon+\rho_a)}\nabla u_\varepsilon.$$

Hence

$$F_\varepsilon=\int_\Omega \frac{1}{2}\frac{\beta}{(u_\varepsilon+\rho_a)}h_\varepsilon\nabla u_\varepsilon+\frac{\eta v}{h_\varepsilon}\,dx. \tag{7}$$

Another important quantity is the load carrying capacity L_ε, which is given by

$$L_\varepsilon=\int_\Omega p_\varepsilon\,dx=\int_\Omega p_a+\beta\ln\frac{w_\varepsilon}{\rho_a}\,dx. \tag{8}$$

For small values of ε (i.e. the roughness scale is much smaller than the global scale) the distance between the surfaces, h_ε, is rapidly oscillating. This means that a direct numerical treatment of (5) will require an extremely fine mesh to resolve the surface roughness. One approach is then to do some type of averaging. The field of mathematics which handles this type of averaging is known as homogenization, (see e.g. Cioranescu & Donato, 1999 or Jikov et al., Jikov, Kozlov, & Oleinik, 1994). The main idea in homogenization is to prove that there exists a p such that $p_\varepsilon \to p$ as $\varepsilon \to 0$ and that p solves a so called homogenized equation. This implies in turn that p may be used as an approximation of p_ε for small values of ε . Moreover, by analyzing the convergence of (p_ε) and (∇p_ε) we are also able to find approximations of L_ε and F_ε for small values of ε. In this paper we derive the homogenized equation corresponding to Eq. (5). We also prove convergence results for the friction force and the load carrying capacity.

Let us now conclude the introduction by giving a short guide to the literature: In the case of an incompressible fluid there are numerous works where homogenization has been used to analyze the effects of surface roughness in hydrodynamic lubrication. Indeed, the Eq. (1) with a constant ρ was homogenized in Wall (2007) by using two-scale convergence, by G-convergence in Chambat, Bayada, and Faure (1988) and by the formal method of multiple scale expansions in Bayada and Faure (1989) and Kane and Bou-Said (2004). The case of several different length scales (both roughness and texture) was analyzed in Almqvist,