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Original article

New implementation of reproducing kernel Hilbert space method for solving a fuzzy integro-differential equation of integer and fractional orders

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ABSTRACT

This paper presents a novel technique for solving two new form of equation with fuzzy and integro-differential equations. The proposed numerical iterative technique is based on the use of the reproducing Kernel theory. Two numerical examples are given to show the effectiveness and performance of the proposed technique. Simulation results are illustrated and comparative studies with past published works to the exact solution from Laplace transform of order integer have been performed to emphasize the simplicity and accuracy of the proposed technique. Moreover, future applications of the proposed technique are also discussed. Numerical experimental results fully support the findings of the proposed analytical approaches.

1. Introduction

The study of Integro-differential Equation has received increasing interest in various physical, biological and engineering sciences (Abu Arqub et al., 2013; Gushing, 2010; Cushing, 2013). For the last two decades, many researchers have paid their attention to the analytical and numerical methods for the solution of Integro-differential Equation. Certain iteration methods subject to periodic boundary conditions have been proposed to solve the Integro-differential Equation (Ben-Zvi et al., 2016; Oza and Callaway, 1987; Heydari et al., 2014; Maayah et al., 2014; Matache et al., 2005; Abbasbandy et al., 2013). The Reproducing Kernel Theory (RKT) has potential applications in integral equations, integro-differential equations, statistics, numerical analysis (Cattani, 2010; Abu Arqub et al., 2012; Jiang and Chen, 2013) among the other numerical and analytical methods. The RKT method has been successfully employed in the concerned literature to investigate certain scientific applications (Yang et al., 2012; Javadi et al., 2014; Abu Arqub, 2015).

Most recently, the authors proposed a reproducing kernel Hilbert space method for solving a system of integro-differential equations of integer order (Abu Arqub, 2015) and fractional order (Bushnaq et al., 2013). In this paper, the authors generalize the idea of RKT method to provide a numerical solution for solving both fuzzy and fractional orders as given in (1). The present work is the extension of the past published works (Yang et al., 2012; Javadi et al., 2014; Abu Arqub, 2015; Bushnaq et al., 2013). To the best of the author's knowledge, the said problem has not been discussed before.

Consider the following form of integro-differential equation of integer and fractional orders:

$$D^{\beta}_{c,0^+}y(x) = g(x, y(x)) + \int_{0^+}^{x} f(t, y(t))dt, y(x_0) = y^F_0, \qquad (1)$$

where $\beta \in (0,1]$ and $x \in [0,1]$, $D^{\alpha}_{c,0^+}$, denotes the left fractional derivatives for Caputo of order β and y^F_0 is a fuzzy value as a triangular number.

It can be observed that Eq. (1) is a general formulation of fuzzy integro-differential equation of fractional orders.

The rest of the paper is organized as follows: Section 2 provides the basic definitions about the integro-differential equations and for clarification of the general formula of the equation fuzzy integro-differential equation of integer and fractional orders, followed by some fuzzy and reproducing kernel definitions in Sections 3 and 4, respectively. Algorithm of the solution based on

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RKT is presented in Section 5. Two numerical examples are given in Section 6. Finally, the paper concludes in Section 7.

2. Caputo and Riemann-Liouville definitions

This section presents some important preliminaries and definitions.

2.1. Caputo definitions

The left and right fractional derivatives by Caputo sense (Manuel and Coito, 2012; Shaher and Odibat, 2007; Caputo, 1967) are defined, respectively as next form:

$$D^{\beta}_{c,a^+}y(x) = \frac{(1)}{\Gamma(\lceil\beta\rceil-\beta)}\int_a^x (x-\tau)^{\lceil\beta\rceil-\beta-1}y^{(\lceil\beta\rceil)}(\tau)d\tau. \quad (2)$$

$$D^{\beta}_{c,b^-}y(x) = \frac{(1)}{\Gamma(\lceil\beta\rceil-\beta)}\int_x^b (\tau-x)^{\lceil\beta\rceil-\alpha-1}y^{(\lceil\beta\rceil)}(\tau)d\tau, \quad (3)$$

where $\Gamma()$ represents the gamma-function, $y^{(\lceil\beta\rceil)}(\tau) = \frac{dy^{\beta}(\tau)}{d\tau^{\lceil\beta\rceil}}$ and $\lfloor\beta\rfloor \leqslant \beta < \lceil\beta\rceil, \beta \in Z^+$, $\lceil\rceil$ is used for the nearest integer number more than β and $\lfloor\rfloor$ is used for the nearest integer number less than β.

2.2. Riemann-Liouville definition

The left and right Riemann-Liouville fractional integral operators of order $\beta > 0$ (Oliveira and Machado, 2014) are defined respectively as next form:

$$I^{\beta}_{RL,a^+}y(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-\tau)^{\beta-1}y(\tau)d\tau \quad (4)$$

$$I^{\beta}_{RL,b^-}y(x) = \frac{1}{\Gamma(\beta)}\int_x^b (\tau-x)^{\beta-1}y(\tau)d\tau. \quad (5)$$

3. Fuzzy definition

In this section, the set of all real numbers is denoted by R and the space of n-dimensional fuzzy number by R^n_F, where $y^F(x): R^n \to [0,1]$

Definition 3.1. Let $y^F(x) \in R^n_F$ and $r \in [0,1]$, and R^n_F denotes the space of n-dimensional fuzzy number. The r-cut off $y^F(x)$ is the crisp set $[y^F(x)]$ that contains all elements with degree in $y^F(x)$ either greater than or equal to r, that is; $[y^F(x)]^r = \{x \in R : y_F(x) \geqslant r\}$, for fuzzy number $u_F(x)$, its r-cut is closed and bounded interval in R and are denoted as follows:

$$[y^F(x)]^r = [y_{1,1r}(x), y_{1,2r}(x)],$$

where,

$$y_{1,1r} = \min\{x : x \in [y^F(x)]^r\}, \quad (6)$$

and

$$y_{1,2r} = \max\{x : x \in [y^F(x)]^r\}, \quad \text{for each} \quad r \in [0,1].$$

For more details, please refer to Buckley and Qu (1991), Arshad and Lupulescu (2011) and Yue et al. (1998).

Definition 3.2. The Triangular and trapezoidal fuzzy numbers respectively (Yue et al., 1998; Ahmad et al., 2013), are defined as follows:

$$y^{TRF}(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leqslant x \leqslant b \\ \frac{c-z}{c-b}, & b \leqslant x \leqslant c \\ 0, & x > c \end{cases} \quad (7)$$

where TRF represents the triangular fuzzy and $y^{TRF}(x) \in R_F$, and its r-cut as follows:

$$[y^{TRF}(x)]^r = [a + r(b-a), c - r(c-b)], \quad \text{for} \quad r \in [0,1] \quad (8)$$

$$y^{TLF}(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leqslant x \leqslant b \\ 1, & b \leqslant x \leqslant c \\ \frac{d-x}{d-c}, & c \leqslant x \leqslant d \\ 0, & x > c \end{cases} \quad (9)$$

where TLF represents the triangular fuzzy and $y^{TLF}(x) \in R_F$, and its r-cut as follows:

$$[y^{TLF}(x)]^r = [a + r(b-a), d - r(d-c)], \quad \text{for} \quad r \in [0,1]. \quad (10)$$

4. Reproducing kernel definitions

Definition 4.1. Let HS $\{y(x)|y(x)$ is a real value function or complex function, $x \in X$ is abstract set$\}$ is a Hilbert space, with inner product $\langle y(x), g(x)\rangle_{HS}$, $(y(x), g(x)) \in HS$

if $\exists U_y(x) \in HS$ $\forall$ fixed $y \in X$, then $U_y(x) \in HS$ and any $y(x) \in HS$ which satisfies the following:

$$\langle y(x), U_y(x)\rangle = y(y), \quad \text{for all} \quad y \in X$$

then,

(i) $U_y(x)$ is a reproducing kernel of HS.
(ii) HS is a reproducing kernel Hilbert space (RKHS).

Definition 4.2 ((Cui and Lin, 2009)). The function space $FS^m_2[a,b]$ is defined as follows:

$$FS^m_2[a,b] = \{y : y^{(i)} \text{ is absolutely continuous}, i = 0,\ldots,m-1, y^{(m)} \in L^2[a,b]\}, \quad (11)$$

where (i) denotes the order of derivative.

Definition 4.3 Cui and Lin, 2009. The inner product in the function space $FS^m_2[a,b]$ for any functions $y(x), v(x) \in FS^1_2[a,b]$ is given by:

$$\langle y, v\rangle_{FS^1_2[a,b]} = u(a)v(b) + \int_a^b y(x)v(x)dx. \quad (12)$$

Definition 4.4 Cui and Lin, 2009. The norm in the function space $FS^m_2[a,b]$ for any functions $y(x), v(x) \in FS^1_2[a,b]$ is defined as follows:

$$||y||_{FS^1_2[a,b]} = \sqrt{\langle y, y\rangle_{FS^1_2[a,b]}} \quad (13)$$

Definition 4.5 Cui and Lin, 2009. The inner product space of $FS^2_2[0,1]$ is defined as: