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On boundedness and compactness of a generalized Srivastava–Owa fractional derivative operator

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Abstract The purpose of this present effort is to define a new fractional differential operator $\mathfrak{T}_z^{\beta,\tau,\gamma}$, involving Srivastava–Owa fractional derivative operator. Further, we investigate some geometric properties such as univalency, starlikeness, convexity for their normalization, we also study boundedness and compactness of analytic and univalent functions on weighted μ-Bloch space for this operator. The method in this study is based on the generalized hypergeometric function.

1. Introduction

The study of fractional operators (integral and differential) plays a vital and essential role in mathematical analysis. Recently, there is a flurry of activity to define generalized differential operators and study their basic properties in a loosely defined area of holomorphic analytic functions in open unit disk. Many authors generalized fractional differential operators on well known classes of analytic and univalent functions to discover and modify new classes and to investigate multi various interesting properties of new classes, for example (see Kiryakova et al., 1998; Dziok and Srivastava, 1999; Srivastava, 2007; Kiryakova, 2010).

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa \tag{1.1}$$

which are analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 1 - f'(0) = 0$, and let $\mathcal{S}$ be the subclass of the $\mathcal{A}$ of the univalent functions in $\mathbb{U}$. Further, a function $f(z) \in \mathcal{S}$ is said to be starlike and convex of order $\lambda (0 \leqslant \lambda < 1)$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \lambda \quad \text{and} \quad \Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \lambda,$$

respectively, these subclasses of $\mathcal{S}$ are denoted by $\mathcal{S}^*$ and $\mathcal{K}$.

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Theorem 1.1 (Bieberbach's Conjecture). *If the function $f(z)$ defined by (1.1) is in the class $\mathcal{S}^*$ then $|a_\kappa| \leqslant \kappa$ for all $\kappa \geqslant 2$ and if it is in the class $\mathcal{K}$ then $|a_\kappa| \leqslant 1$ for all $\kappa \geqslant 2$ (Duren, 1983).*

For $f(z)$ given by (1.1) and $g(z) = z + \sum_{\kappa=2}^{\infty} b_\kappa z^\kappa$, the convolution (or Hadamard product) $f * g$ is defined by

$$f * g(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa z^\kappa. \tag{1.2}$$

The operator $\mathcal{O}_z^{\beta,\tau}$ is defined in terms of Riemann–Liouville fractional differential operator $\mathcal{D}_z^{\beta-\tau}$ as

$$\mathcal{O}_z^{\beta,\tau} f(z) = \frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} \mathcal{D}_z^{\beta-\tau} z^{\beta-1} f(z) \quad z \in \mathbb{U}. \tag{1.3}$$

This operator is given by Tremblay (1974). Recently, Ibrahim and Jahangiri (2014) extended Tremblay's operator in terms of Srivastava–Owa fractional derivative of $f(z)$ of order $(\beta-\tau)$ and is defined as follows

$$\mathfrak{T}_z^{\beta,\tau} f(z) = \frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} D_z^{\beta-\tau} z^{\beta-1} f(z) \tag{1.4}$$

Often, the generalized fractional differential operators and their applications associated with special functions, Dziok and Srivastava (1999), defined a linear operator as a Hadamard product with an arbitrary ${}_pF_q$-function $p \leqslant q+1$, several authors interested Dziok–Srivastava operator as well as Srivastava–Wright operator, which is defined and investigated by Srivastava (2007). Recently Kiryakova (2011), considered those operators and studied their criteria univalence properties in the class $\mathcal{A}$.

Definition 1.1. The Fox–Wright ${}_p\Psi_q$ generalization of the hypergeometric ${}_pF_q$ function is defined as:

$${}_p\Psi_q[z] = {}_p\Psi_q\left[\begin{matrix}(a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{\kappa=0}^{\infty} \frac{\Pi_{j=1}^{p}\Gamma(a_j+\kappa A_j)}{\Pi_{j=1}^{q}\Gamma(b_j+\kappa B_j)(1)_\kappa} z^\kappa, \tag{1.5}$$

where a_j, b_j are parameters in complex plane $\mathbb{C}$. $A_j > 0,\ B_j > 0$ for all $j = 1, \ldots, q$ and $j = 1, \ldots, p$, such that $0 \leqslant 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_i$ for fitting values $|z| < 1$ and it is well know that

$${}_p\Psi_q\left[\begin{matrix}(a_j, 1)_{1,p} \\ (b_j, 1)_{1,q}\end{matrix}\middle| z\right] = \Delta^{-1}{}_pF_q(a_j, b_j; z), \quad \text{with} \quad \Delta = \frac{\Pi_{j=1}^{q}\Gamma(b_j)}{\Pi_{j=1}^{p}\Gamma(a_j)} \tag{1.6}$$

where ${}_pF_q$ is the generalized hypergeometric function and $(v)_\kappa = \Gamma(v+\kappa)/\Gamma(v)$ is the *Pochhammer* symbol (see Kilbas et al., 2006).

Remark 1.1. By usage the Hadamard product technique, Srivastava (2007) provided families of analytic and univalent functions associated with the Fox–Wright generalized hypergeometric functions ${}_p\Psi_q$ in the open unit disk $\mathbb{U}$.

In the present paper the new generalized fractional differential operator $\mathfrak{T}_z^{\beta,\tau,\gamma}$ of analytic function is defined. Also, the univalence properties of the normalization generalized operator are investigated and proved. Further, the boundedness and compactness of this operator are studied.

2. Background and results

In this section, we consider the generalized type fractional differential operator and then we determine the generalized fractional differential of some special functions. For this main purpose, we begin by recalling the Srivastava–Owa fractional derivative operators of $f(z)$ of order β defined by

$$D_z^\beta f(z) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dz} \int_0^z f(\zeta)(z-\zeta)^{-\beta} d\zeta, \tag{2.1}$$

where $0 \leqslant \beta < 1$, and the function $f(z)$ is analytic in simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\beta}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$ (see Owa, 1978; Owa and Srivastava, 1987). Then under the conditions of the above definition the Srivastava–Owa fractional derivative of $f(z) = z^\kappa$ is defined by

$$D_z^\beta\{z^\kappa\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\rho+1)} z^{\kappa-\beta}.$$

The theory of fractional integral and differential operators has found significant importance applications in various areas, for example (see Dziok and Srivastava, 2003). Recently, many mathematicians have developed various generalized fractional derivatives of Srivastava–Owa type, for example, (Srivastava et al., 2010 and Kiryakova, 2011). Further, we consider a generalized Srivastava–Owa type fractional derivative formulas which recently appeared.

Definition 2.1 (Ibrahim, 2011). The generalized Srivastava–Owa fractional derivative of $f(z)$ of order β is defined by

$$\mathcal{D}_z^{\beta,\gamma} f(z) := \frac{(\gamma+1)^\beta}{\Gamma(1-\beta)} \frac{d}{dz} \int_0^z \left(z^{\gamma+1} - \zeta^{\gamma+1}\right)^{-\beta} \zeta^\gamma f(\zeta) d\zeta, \tag{2.2}$$

where $0 \leqslant \beta < 1$, $\gamma > 0$ and $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin. In particular, the generalized Srivastava–Owa fractional derivative of function $f(z) = z^\kappa$ is defined by

$$\mathcal{D}_z^{\beta,\gamma}\{z^\kappa\} = \frac{(\gamma+1)^{\beta-1}\Gamma\left(\frac{\kappa}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa}{\gamma+1}+1-\beta\right)} z^{(1-\beta)(\gamma+1)+\kappa-1}.$$

Now, we present a new generalized fractional differential operator $\mathfrak{T}_z^{\beta,\tau,\gamma}$ as follows:

Definition 2.2. The generalized fractional differential of $f(z)$ of two parameters β and τ is defined by

$$\mathfrak{T}_z^{\beta,\tau,\gamma} f(z) := \frac{(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)\Gamma(1-\beta-\tau)} \left(z^{1-\tau}\frac{d}{dz}\right) \int_0^z \frac{\zeta^{\gamma+\beta-1} f(\zeta)}{\left(z^{\gamma+1} - \zeta^{\gamma+1}\right)^{\beta-\tau}} d\zeta, \tag{2.3}$$

$(\gamma \geqslant 0; \quad 0 < \beta \leqslant 1; \quad 0 < \tau \leqslant 1; \quad 0 \leqslant \beta - \tau < 1)$,

where the function $f(z)$ is analytic in simple-connected region of the complex z-plane $\mathbb{C}$ containing the origin.

Remark 2.1. For $f(z) \in \mathcal{A}$, we have

i. when $\gamma = 0$ in (2.3), is reduced to the classical known one (1.4) and