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q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions

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Abstract In this paper, we prove the correct q-Hermite–Hadamard inequality, some new q-Hermite–Hadamard inequalities, and generalized q-Hermite–Hadamard inequality. By using the left hand part of the correct q-Hermite–Hadamard inequality, we have a new equality. Finally using the new equality, we give some q-midpoint type integral inequalities through q-differentiable convex and q-differentiable quasi-convex functions. Many results given in this paper provide extensions of others given in previous works.

1. Introduction

The study of calculus without limits is known as quantum calculus or q-calculus. The famous mathematician Euler initiated the study q-calculus in the eighteenth century by introducing the parameter q in Newton's work of infinite series. In early twentieth century, Jackson (1910) has started a symmetric study of q-calculus and introduced q-definite integrals. The subject of quantum calculus has numerous applications in various areas of mathematics and physics such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions, quantum theory, mechanics and in theory of relativity. This subject has received outstanding attention by many researchers and hence it is considered as an in-corporative subject between mathematics and physics. Interested readers are referred to Ernst (2012), Gauchman (2004), and Kac and Cheung (2001) for some current advances in the theory of quantum calculus and theory of inequalities in quantum calculus.

In recent articles, Tariboon and Ntouyas (2013, 2014) studied the concept of q-derivatives and q-integrals over the intervals of the form $[a, b] \subset \mathbb{R}$ and settled a number of quantum analogs of some well-known results such as Holder inequality, Hermite–Hadamard inequality and Ostrowski inequality, Cauchy–Bunyakovsky–Schwarz, Gruss, Gruss–Cebysev and other integral inequalities using classical convexity. Also, Noor et al. (2015), Noor et al. (2015),

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Sudsutad et al. (2015), and Zhuang et al., 2016, have contributed to the ongoing research and have developed some integral inequalities which provide quantum estimates for the right part of the quantum analog of Hermite–Hadamard inequality through q-differentiable convex and q-differentiable quasi-convex functions.

Let real function f be defined on some non-empty interval I of real line $\mathbb{R}$. The function f said to be convex on I, if the inequality

$$f(ta+(1-t)b) \leqslant tf(a)+(1-t)f(b)$$

holds for all $a,b \in I$ and $t \in [0,1]$. The function f said to be quasi-convex on I, if the inequality

$$f(ta+(1-t)b) \leqslant \sup\{f(a),f(b)\}$$

holds for all $a,b \in I$ and $t \in [0,1]$.

Kırmacı (2004) obtained inequalities for differentiable convex mappings which are connected with midpoint type inequality, Alomari et al. (2009) obtained inequalities for differentiable quasi-convex mappings which are connected with midpoint type inequality. They used the following lemma to prove their theorems.

Lemma 1 Kırmacı (2004). *Let $f: I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a<b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)$$
$$= b-a\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt\right] \tag{1.1}$$

2. Preliminaries and definitions of q-calculus

Throughout this paper, let $a<b$ and $0<q<1$ be a constant. The following definitions and theorems for q-derivative and q-integral of a function f on $[a,b]$ are given in Tariboon and Ntouyas (2013, 2014).

Definition 2. For a continuous function $f:[a,b] \to \mathbb{R}$ then q-derivative of f at $x \in [a,b]$ is characterized by the expression

$${}_aD_q f(x) = \frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{2.1}$$

Since $f:[a,b] \to \mathbb{R}$ is a continuous function, thus we have ${}_aD_q f(a) = \lim_{x\to a} {}_aD_q f(x)$. The function f is said to be q-differentiable on $[a,b]$ if ${}_aD_q f(t)$ exists for all $x \in [a,b]$. If $a=0$ in (2.1), then ${}_0D_q f(x) = D_q f(x)$, where $D_q f(x)$ is familiar q-derivative of f at $x \in [a,b]$ defined by the expression (see Kac and Cheung, 2001)

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad x \neq 0. \tag{2.2}$$

Definition 3. Let $f:[a,b] \to \mathbb{R}$ be a continuous function. Then the q-definite integral on $[a,b]$ is delineated as

$$\int_a^x f(t)\,{}_ad_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a) \tag{2.3}$$

for $x \in [a,b]$.

If $a=0$ in (2.3), then $\int_0^x f(t)\,{}_0d_q t = \int_0^x f(t)d_q t$, where $\int_0^x f(t)d_q t$ is familiar q-definite integral on $[0,x]$ defined by the expression (see Kac and Cheung, 2001)

$$\int_0^x f(t)\,{}_0d_q t = \int_0^x f(t)d_q t = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x). \tag{2.4}$$

If $c \in (a,x)$, then the q-definite integral on $[c,x]$ is expressed as

$$\int_c^x f(t)\,{}_ad_q t = \int_a^x f(t)\,{}_ad_q t - \int_a^c f(t)\,{}_ad_q t. \tag{2.5}$$

Theorem 4 Tariboon and Ntouyas (2014, Theorem 3.2). *Let $f:[a,b] \to \mathbb{R}$ be a convex continuous function on $[a,b]$ and $0<q<1$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,{}_ad_q t \leqslant \frac{qf(a)+f(b)}{1+q}. \tag{2.6}$$

Kunt and İşcan (2016) give the following example to prove that the left hand side of (2.6) is not correct:

Example 5. Let $[a,b]=[0,1]$. Then the function $f(t)=1-t$ is a convex continuous function on $[0,1]$. Therefore the function f satisfies Theorem 4 assumptions. Then, from the inequality (2.6) the following inequality must be hold for all $q \in (0,1)$

$$f\left(\frac{0+1}{2}\right) \leqslant \frac{1}{1-0}\int_0^1 f(t)\,{}_0d_q t$$

$$1-\frac{1}{2} \leqslant (1-q)\sum_{n=0}^{\infty} q^n(1-q^n)$$

$$\frac{1}{2} \leqslant (1-q)\left(\frac{1}{1-q}-\frac{1}{1-q^2}\right)$$

Then we have

$$\frac{1}{2} \leqslant \frac{q}{1+q}. \tag{2.7}$$

If we choose $q=\frac{1}{2}$ in (2.7) we have the following contradiction

$$\frac{1}{2} \leqslant \frac{1}{3}.$$

It means that the left hand side of (2.6) is not correct.

In the next section we give the correct q-Hermite–Hadamard inequality, some q-Hermite–Hadamard inequalities, and generalized q-Hermite–Hadamard inequality.

3. q-Hermite–Hadamard inequalities

In this section we prove q-Hermite–Hadamard inequality and varieties of q-Hermite–Hadamard inequalities.