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Some new Hermite-Hadamard type inequalities for *MT*-convex functions on differentiable coordinates

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ABSTRACT

In this paper, we introduce the notion of *MT*-convex functions on co-ordinates and establish some new integral inequalities of Hermite-Hadamard type for *MT*-convex functions on co-ordinates on a rectangle Δ in the plane $\mathbb{R}^2$.

1. Introduction

Let us recall some definitions of various convex functions that are known in the literature.

Definition 1.1 (*Guo et al., 2016; Sarikaya et al., 2016*). A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval I, if for all $x, y \in I$ and $t \in (0,1)$ it satisfies the following inequality:

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y). \tag{1.1}$$

Definition 1.2 (*Tunç et al., 2013; Park, 2015*). A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *MT*-convex on I, if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ it satisfies the following inequality:

$$f(tx + (1-t)y) \leqslant \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.2}$$

Example of such functions are:

(1) The functions $f, g : (1, \infty) \to \mathbb{R}$, where

$$f(x) = x^p \quad \text{and} \quad g(x) = (1+x)^p, \quad p \in \left(0, \frac{1}{1000}\right)$$

(2) The function $h : [1, \frac{3}{2}] \to \mathbb{R}$, where

$$h(x) = (1+x^2)^q, \quad q \in \left(0, \frac{1}{1000}\right).$$

Notice that these functions are not convex.

Definition 1.3 Guo et al., 2016. If $(X, \mathcal{A})$ is a measurable space, then $f : X \to \mathbb{R}$ is measurable if $f^{-1}(B) \in \mathcal{A}$ for every Borel set $B \in \mathcal{B}(\mathbb{R})$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable if $f^{-1}(B)$ is a Lebesgue measurable subset of $\mathbb{R}^n$ for every Borel subset B of $\mathbb{R}$.

Let us now consider a formal definition for co-ordinated convex functions:

Definition 1.4 (*Dragomir et al., 2000; Dragomir, 2001*). A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on $\Delta = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ with $a < b$ and $c < d$ if for all $t, \lambda \in (0,1)$ and $(x,y), (z,w) \in \Delta$ satisfies the following inequality:

$$\begin{aligned} f(tx + (1-t)z, \lambda y + (1-\lambda)w) \leqslant\; & t\lambda f(x,y) + t(1-\lambda) f(x,w) \\ & + (1-t)\lambda f(z,y) + (1-t)(1-\lambda) f(z,w). \end{aligned} \tag{1.3}$$

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Definition 1.5 Samko et al., 1993. The incomplete beta function is defined by

$$B_x(a,b) = \int_0^x z^{a-1}(1-z)^{b-1}dz, \quad a,b>0.$$

For $z=1$, the incomplete beta function coincides with the complete beta function.

Throughout this paper we denote by $L_1(\Delta)$ the set of all Lebesgue integrable functions on Δ as indicated by the authors in Guo et al. (2016). Some integral inequalities of Hermite-Hadamard type for co-ordinated convex functions on the rectangle in the plane $\mathbb{R}^2$ may be recited as follows:

Theorem 1.1 (*Dragomir et al., 2000; Dragomir, 2001, Theorem 2.2*). Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ be convex on the co-ordinates on Δ with $a<b$ and $c<d$. Then

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leqslant \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \right.\\ &\quad \left. +\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\ &\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dy\,dx\\ &\leqslant \frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x,c)dx+\int_a^b f(x,d)dx\right)\right.\\ &\quad \left.+\frac{1}{d-c}\int_c^d\left(\int_c^d f(a,y)dx+\int_c^d f(b,y)dx\right)\right]\\ &\leqslant \frac{1}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)]. \end{aligned}$$

Theorem 1.2 Guo et al., 2015, Theorem 2.1. Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partial differentiable mapping on Ω^o (the interior of Ω) and let $\Delta=[a,b]\times[c,d]\subseteq\Omega^o$ with $a<b, c<d$ and $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is convex on the co-ordinates on Δ and $q\geqslant 1$, then the following inequality holds:

$$\begin{aligned}|I(f)| &\leqslant \frac{1}{4}\left(\frac{1}{9}\right)^{\frac{1}{q}}\{g_q(1,2,2,4)+g_q(4,2,2,1)+g_q(2,1,4,2)\\ &\quad +g_q(2,4,1,2)\},\end{aligned}$$

where

$$\begin{aligned} I(f) &= \frac{16}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right.\\ &\quad \left.-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dx\,dy\right],\end{aligned}$$

and

$$\begin{aligned} g_q(r_1,r_2,r_3,r_4) &= \left[r_1|f_{xy}(a,c)|^q+r_2|f_{xy}(a,d)|^q+r_3|f_{xy}(b,c)|^q\right.\\ &\quad \left.+r_4|f_{xy}(b,d)|^q\right]^{\frac{1}{q}}.\end{aligned}$$

For more information on integral inequalities of the Hermite-Hadamard type for various kinds of convex functions, the reader is referred to the recently published papers (Park, 2013; Guo et al., 2016; Meftah and Boukerrioua, 2015; Xi and Qi, 2015; Bai et al., 2016), and the closely related references therein.

In this paper, we will establish more integral inequalities of the Hermite-Hadamard type for *MT*-convex functions on the co-ordinates on a rectangle Δ in the plane $\mathbb{R}^2$.

2. A definition and a lemma

Motivated by Definitions 1.1 and 1.3, we introduce the notion of "co-ordinated *MT*-convex function".

Definition 2.1. We say that a function $f:\Delta\to\mathbb{R}$ is *MT*-convex on the co-ordinates on $\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2$ with $a<b$ and $c<d$, if it is nonnegative and for all $t,\lambda\in(0,1)$ and $(x,y),(z,w)\in\Delta$ it satisfies the following inequality:

$$\begin{aligned} f(tx+(1-t)z,\lambda y+(1-\lambda)w) &\leqslant \frac{\sqrt{t\lambda}}{4\sqrt{(1-t)(1-\lambda)}}f(x,y)\\ +\frac{\sqrt{t(1-\lambda)}}{4\sqrt{\lambda(1-t)}}f(x,w)+\frac{\sqrt{\lambda(1-t)}}{4\sqrt{t(1-\lambda)}}f(z,y)&+\frac{\sqrt{(1-t)(1-\lambda)}}{4\sqrt{t\lambda}}f(z,w).\end{aligned} \tag{2.1}$$

Now, we give an example to show that a function can be *MT*-convex on the co-ordinates on Δ without being convex on the co-ordinates on Δ. The function $f(x,y):(1,\infty)\times(1,\infty)\to\mathbb{R}$, where

$$f(x,y)=x^c+y^c \quad \text{for} \quad c\in\left(0,\frac{1}{1000}\right)$$

is *MT*-convex on the co-ordinates on $\Delta=(1,\infty)\times(1,\infty)$ while this is not convex on the co-ordinates on Δ.

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partial differentiable mapping on Ω^o and let $\Delta=[a,b]\times[c,d]\subseteq\Omega^o$ with $a<b,c<d$ and $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$. Then the following equality holds:*

$$\begin{aligned} I(f) :&= \frac{16}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right.\\ &\quad \left.-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dx\,dy\right]\\ &= \int_0^1\int_0^1 t\lambda f_{xy}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)dt\,d\lambda\\ &\quad +\int_0^1\int_0^1 t\lambda f_{xy}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)dt\,d\lambda\\ &\quad -\int_0^1\int_0^1 t\lambda f_{xy}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\left(1-\frac{\lambda}{2}\right)c+\frac{\lambda}{2}d\right)dt\,d\lambda\\ &\quad -\int_0^1\int_0^1 t\lambda f_{xy}\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)dt\,d\lambda. \end{aligned} \tag{2.2}$$

Proof. By integration by parts, we have

$$\begin{aligned} &\int_0^1\int_0^1 t\lambda f_{xy}\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)dt\,d\lambda\\ &=\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_0^1 f\left(\frac{a+b}{2},\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)d\lambda\right.\\ &\quad\left.-\int_0^1 f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{c+d}{2}\right)dt+\int_0^1\int_0^1 f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b,\frac{\lambda}{2}c+\left(1-\frac{\lambda}{2}\right)d\right)dt\,d\lambda\right]\\ &=\frac{4}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2}{d-c}\int_{\frac{c+d}{2}}^d f\left(\frac{a+b}{2},y\right)dy\right.\\ &\quad\left.-\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f\left(x,\frac{c+d}{2}\right)dx+\frac{4}{(b-a)(d-c)}\int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b f(x,y)dx\,dy\right]. \end{aligned}$$

Similarly, we find