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Existence and qualitative properties of solutions for Choquard equations with a local term

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In this paper, a nontrivial solution $u \in H^1(\mathbb{R}^N)$ to the autonomous Choquard equation with a local term

$$-\Delta u + \lambda u = (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u \quad \text{in } \mathbb{R}^N$$

is obtained, where $N \geq 3$, $\alpha \in (0, N)$, $\lambda > 0$ is a constant, I_α is the Riesz potential, $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $q \in (2, 2^* = \frac{2N}{N-2})$. Under some further assumptions on p and q, the regularity and the Pohožaev identity of the solution are established, and then it is shown that the obtained solution is a groundstate of mountain pass type. Moreover, the positivity and symmetry of the groundstate are also considered. By using the results obtained for the autonomous equation, a positive groundstate solution for the nonautonomous equation

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u \quad \text{in } \mathbb{R}^N$$

is also found under some assumptions on $V(x)$.

1. Introduction and main results

We are interested in the stationary Choquard equation with a local term

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, I_α is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}}$$

with Γ denoting the Gamma function(see [1], P.19), $V(x)$, p and q will be defined later.

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Eq. (1.1) can be viewed as a local nonlinear perturbation of

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

For $N = 3,\ p = 2$ and $\alpha = 2$, (1.2) was investigated by Pekar in [2] to study the quantum theory of a polaron at rest. In [3], Choquard applied it as an approximation to Hartree–Fock theory of one component plasma. It also arises in multiple particles systems [4] and quantum mechanics [5]. There are many papers devoted to the existence and multiplicity of solutions of (1.2) and their qualitative properties. See the survey paper [6] and the references therein. To study problem (1.2) variationally the well-known Hardy–Littlewood–Sobolev inequality is the starting point. Usually, $1 + \frac{\alpha}{N}$ is called the lower critical exponent and $\frac{N+\alpha}{N-2}$ is the upper critical exponent for the Choquard equation. We also recall that $2^* = 2N/(N-2)$ is the critical Sobolev exponent. For the subcritical autonomous case $p \in \left(1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2}\right)$ and $V(x) \equiv 1$, V. Moroz and J. Van Schaftingen [7] established the existence, qualitative properties and decay estimates of groundstate solutions of (1.2). They also obtain some nonexistence results under the range

$$p \geq \frac{N+\alpha}{N-2} \quad \text{or} \quad p \leq 1 + \frac{\alpha}{N}.$$

However, to the best of our knowledge, a few have been done for Eq. (1.1) concerning the existence of nontrivial solutions and their qualitative properties. When (1.1) is autonomous, i.e. $V(x) \equiv 1$, J. Chen and B. Guo [8] obtained a groundstate of (1.1) by using Nehari manifold method for $N = 3,\ p = 2,\ \alpha \in (0,1)$ and $4 \leq q < 6$. G. Vaira [9] obtained a positive radial groundstate when $N = 3,\ \alpha = 2,\ p = 2,\ q \in (2,6)$, and G. Vaira [10] further studied the nondegeneracy of the radial groundstate for the special case $q = 3$. Y. Ao [11] gave a nontrivial solution of (1.1) if p is the upper critical exponent $\frac{N+\alpha}{N-2}$ and J. Seok [12] constructed a radially symmetric solution of (1.1) for the critical case $q = 2^*$. The authors in [13] and [14] considered the groundstate and mountain pass type solution for the two dimensional case $N = 2$, $I_\alpha(x) = -\log|x|$ and $p = 2$. For the nonautonomous equation, $V(x)$ depends on x in (1.1), most results are for $N = 3,\ \alpha = 2$ and $p = 2$. See G. Vaira [9] for the existence of a positive groundstate and G. Vaira [10] for the existence of a positive bound state solution when groundstates may not exist. For more results, see [15–20] for the corresponding Schrödinger–Poisson system.

More recently, J. Van Schaftingen and J. Xia [21] considered the lower critical problem

$$-\Delta u + u = (I_\alpha * |u|^{1+\frac{\alpha}{N}})|u|^{\frac{\alpha}{N}-1}u + f(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

When $f(x,u) = f(u)$ satisfies some assumptions, they studied the existence and symmetry of groundstate of (1.3). On the basis of this, they further derived a groundstate of (1.3) for the nonautonomous case $f(x,u) = K(x)|u|^{q-2}u$ with $q \in (2, 2 + \frac{4}{N})$ and $K(x) \in L^\infty(\mathbb{R}^N)$ satisfying

$$\inf_{x \in \mathbb{R}^N} K(x) = K_\infty = \lim_{|x| \to \infty} K(x) > 0.$$

Motivated by [22,23] and [21], in this paper, we will study the existence and qualitative properties of groundstates of (1.1) for the subcritical case

$$\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2} \text{ and } q \in (2, 2^*). \tag{1.4}$$

At the first step, we study the autonomous nonlinear Choquard equation

$$-\Delta u + \lambda u = (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $\lambda > 0$ is a constant. By (1.4), the Hardy–Littlewood–Sobolev inequality and the Sobolev embedding theorem, the functional $J_\lambda :\ H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda |u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q \tag{1.6}$$